THREE TYPES OF LINEAR THEORIES FOR ATOMIZING LIQUIDS

S.P. Lin* and Z.L. Wang
Department of Mechanical and Aeronautical Engineering
Clarkson University
Potsdam, NY 13699, USA

Abstract
There are three types of linear stability theories which are currently being used to predict the onset of breakup of liquid jets or sheets. Temporal theory which is most commonly used, because of its simplicity, assumes that the disturbance responsible for the breakup grows temporally at the same rate everywhere in space. A less commonly used spatial theory assumes the disturbance grows in space, because the breakup appears to take place in the region downstream of the location where the liquid is introduced. The most complete theory is that of spatio-temporal instability. This theory has not yet been applied widely because of its mathematical and numerical complexity. It is demonstrated here with an example that a flow may be predicted to be neutral according to pure spatial or pure temporal theory, while it is actually stable according to the spatio-temporal theory. The prediction of the latter theory is shown to agree with the numerical solution of the initial value problem.

*Corresponding author
INTRODUCTION

The onset of breakup of a plane liquid sheet has been investigated by Squire [1], Fraser et al. [2], Hagerty and Shea [3]. These authors assumed that the disturbance which causes the sheet breakup is periodic in space and grows temporally everywhere in space at the same rate. The linear Stability theory based on this assumption is called pure temporal theory. The application of the temporal theory to various problems of jet breakup has been recently reviewed by Yoon and Heister [4]. The temporal theory predicts that a plane liquid sheet of an inviscid fluid is unstable in We>1, where We=$\rho U^2 h/S$ is the Weber number, $\rho$, U, h, and S being, respectively the density, velocity, and the half sheet thickness. The disturbance in the temporal theory does not grow in space, and thus its temporal growth rate cannot be compared directly with the experimentally observed spatial growth. However the spatial growth rate may be inferred in some cases by multiplying the temporal growth rate with the group velocity to yield approximately the growth rate of the disturbance as they are convected as a group in the downstream direction. This approach has been applied by Reitz and Braco [5] to estimate the jet breakup length. On the other hand when We<1, the temporal theory predicts that a breakup cannot be initiated by small disturbances in the liquid sheet. This is contrary to the experimental observation of Brown [6] and Taylor [7]. Both of them observed the breakup to take place in We<1. This paradox is resolved by de Luca and Costa [8]. They consider the disturbance to be growing both spatially and temporally. They were able to predict that when We<1, the disturbance actually grows in time as it is convected simultaneously in the downstream as well as upstream direction. Their results agree qualitatively with the experiments of Brown and Taylor. The distinction between the temporal and spatio-temporal theories has been further expounded by Lin [9].

There is a third type of linear stability theory which considers the disturbance to oscillate in time and grow in space. This type of theory is termed pure spatial theory. The danger of a naïve use of this type of theory is already pointed out by Drazin and Reid [10] in connection with shallow water wave motion. The use of the spatial theory and the other two types of theories to the liquid sheet will be contrasted in this paper. The theoretical predictions mentioned in the previous paragraph are all based on the large time asymptotic behavior of the disturbance. The physical significance of the large time asymptotic results must be properly understood. A disturbance may grow initially before decaying to zero as time goes to infinity as predicted by the linear theory. If the initial growth is so rapid that nonlinear effects become important in a very short time, then the large time asymptotic linear predictions become insignificant. The initial transient growth of the varicose mode of disturbance will be related with its large time behavior in the plane liquid sheet. This particular example is chosen here because the varicose mode was not considered in the writer’s previous work on the comparison between the temporal and spatio-temporal theories [9]. The pure spatial theory will be also applied to the same example to Point out its pitfall. The objective of this paper is to illustrate how the linear theories may be properly applied to the problem of atomization and sprays.

LINEAR STABILITY THEORIES

Taylor [7] derived the following partial differential equation for the onset of long wavelength varicose disturbances in a plane liquid sheet,

$$(\delta_t + \delta_x)^2 F(t,x) + W e^{-1} \delta_{xxx} F = 0.$$  \hspace{1cm} (1)

Eq. (1) was derived from Euler’s equation of fluid motion and the corresponding boundary conditions, neglecting the effect of gravity and the surrounding air. In Eq. (1), t is the time non-dimensionalized with $h/U$, h and U being respectively the uniform half thickness and velocity of the flowing liquid sheet; x is the distance in the flow direction normalized with h, and F is the displacement of the free surface by the disturbance from the unperturbed free surface in the y-direction which is perpendicular to x. The displacement is also normalized with h. In arriving at Eq. (1), the velocity and pressure are normalized respectively with U and $U^2$.

Taking the double Fourier transform of Eq. (1) with respect to t and x, one has

$$D = [W e^{-k^2} - (k - \alpha)^2] f = \delta(x_0,t_0),$$  \hspace{1cm} (2)

where

$$f = \int |F(x,t)| e^{i(kx - \alpha t)} dx dt,$$  \hspace{1cm} (3)

and $\delta(x_0,t_0)$ arises from the Dirac delta displacement introduced on the free surface at $x = x_0$, and $t = t_0$. The wave number k and the wave frequency $\omega$ in Eq. (1) are defined by $k = K/h$ and $\omega = \Omega/(U/h)$ were K and $\Omega$ are the dimensional wave number and frequency respectively. The physical free surface position can be obtained by the inverse Fourier transform of f

$$F = \frac{1}{2\pi} \int f e^{i(kx - \alpha t)} dkd\omega.$$  \hspace{1cm} (4)

It follows from Eq. (3) that

$$F = \frac{1}{2\pi} \int \frac{\delta(x_0,t_0)}{D} e^{i(kx - \alpha t)} dkd\omega.$$  \hspace{1cm} (5)

Note that the Fourier integral (5) can be evaluated, since D’ approaches zero at least as fast as k and $\omega$ approach infinity. Moreover the major contribution of the integral comes from those disturbances which satisfy
The solution of the dispersion equation (6) for a given parameter We, characterizes the natural frequency and wavelength of the disturbance in the liquid sheet.

**TEMPORAL THEORY**

In the temporal theory, a normal mode $f=\exp[\pm ikx+\omega t]$ is considered $k$ is real but $\omega$ is complex. The real part $\omega_r$ gives the wave frequency and the imaginary part $\omega_i$ gives the exponential temporal growth rate, if positive. If $\omega_i$ is negative the sheet is deemed stable. If it is zero the sheet can sustain an undamped neutral oscillation with corresponding frequency $\omega_r$. The solution of (6) gives

$$\omega_r = 0,$$

$$\omega_i = k_r \pm We^{-1/2} k_r.$$  \hspace{1cm} (8)

Hence the disturbance amplitude cannot grow or decay in time according to Eq. (7). The disturbance amplitude remains everywhere the same by the apriori assumption of the temporal theory. Eq. (8) states that the neutrally oscillating disturbance of a wave number $kr$ has a phase velocity $c=\omega_r/k_r=\pm We^{-1/2} k_r$. The upstream and downstream branches meet at $Z=0$ at the onset. As we reduce $k$ below $k_{w_0}$, the upstream branch of the amplification curve, $u$, approaches the dashed line from above the $\omega$ axis. The lower part of curve $d$ approaches the line $k=0$. Any point on these curves is an isolated pole in the integrand of Eq. (5). As the linear theory is applicable only at the onset of instability, we must reduce the value of $\omega_i$ from 1 on the curves $d$ and $u$ gradually to $\omega_i = 0$ at the onset. As we reduce $\omega_i$ to zero, the upper part of curve $d$ approaches the line $k = -W_0^2/2$ and the lower part of curve $d$ approaches the $k_r$ axis from below. The entire curve $d$ does not cross over the $k_r$ axis as indicated by the dotted line. Similarly as $\omega_i$ is reduced from 1 to zero the upstream branch of the amplification curve, $u$, approaches the dashed line from below the $k_r$ axis, but does not cross over to the upper plane. The upstream and downstream branches meet at $(k_{\omega_0}, k_{\omega_0}) = (-W_0^2/2, 0)$. At this point $D = \partial F / \partial k = 0$, but $\partial F / \partial \omega \neq 0$. Hence this point is a singularity higher than a simple pole. However this point is not a pinch point which signals absolute instability, since the down-
stream and upstream branches of the amplification curves do not cross over the real k-r-axis [11,12].

Nevertheless the singularity is of a higher order at which, to the lowest order approximation, the Taylor’s series expansion of D gives the relation \((k - k_0)^2 \sim (\omega - \omega_0)\). This allows us to obtain the large time asymptotic behavior of F from a general formula [11,12],

\[
\lim_{t \to \infty} F(t^{-1}) \exp\left[ x = U_{g0}(t^{-i} \omega_0 t) \right],
\]

where \(U_{g0} = \omega_0 / d k_0 \) is the group velocity, \(v = n(p+1)/(p+2)\) in which n and p are respectively the powers of the first nonvanishing terms of \((k - k_0)\) and \((\omega - \omega_0)\) in the Taylor series expansion of D. Thus \((k - k_0)^{p+2} \sim (\omega - \omega_0)^n\) in D. In this example p=0 and n=1, and thus \(v = 1/2\). Hence

\[
\lim_{t \to \infty} F(t^{-1/2}) \exp\left[ x = U_{g0}(t^{-i} \omega_0 t) \right],
\]

where \(\omega_0 = -WE^2 / 4\) and \(U_{g0} = 1\) according to Eq. (11) evaluated at \(k_0 = 0\).

The singularity corresponding to the branch with negative coefficient of \(W_e\) in Eqs. (11) and (12) is located at \((k_{m0}, k_{n0}) = (WE^2 / 2, 0)\) which is the mirror image about the k-i-axis of the previous singular point \((-WE^2 / 2, 0)\). The amplification curves d and u for \(\omega_1 = 1\) and \(\omega_0 = 0\) are also the mirror images of the previous case. The same argument carries over from the previous case and leads to the same asymptotic behavior given by Eq. (13) except the sign of \(\omega_0\) is changed.

The large time behavior given in Eq. (13) will be confirmed by the initial value solution given in the next section.

**INITIAL VALUE PROBLEM**

The initial value solution of Eq. (1) is carried out by use of the second order difference scheme. Eq. (1) is approximated by

\[
\frac{F_{i+1} - 2F_i + F_{i-1}}{(\Delta t)^2} + \frac{F_{i+1} - 2F_i + F_{i-1}}{(\Delta x)^2} - \frac{F_{i+1} - F_{i-1}}{(\Delta x)(\Delta t)} = -WE^2 (\Delta x)^{-4}[F_{i+2} - 4F_{i+1} + 6F_i - 4F_{i-1} + F_{i-2}],
\]

(14)

The initial boundary conditions are given by

\[
F(0,x) = 0.05[\cos \pi(x/0.1) + 1], \quad -0.1 < x < 0.1;
\]

\[
F(0, x) = 0, \quad -0.1 > x > 0.1;
\]

\[
\partial_x F(0, x) = 0, \quad F(t, \pm \infty) = 0.
\]

For a stable solution at any n-th time step of magnitude \(\Delta t\), we chose the spatial distance \(\Delta x\) between any i-th nodal point with its neighboring nodal point to satisfy the condition \(W_e^2 \Delta t(\Delta x)^2 \leq 0.5\).

Fig. 2 shows the time evolution of the initial disturbance given by the set of equations (15). It is seen that the initial disturbances spread in space and decay in time as they are convected as downstream by the sheet flow. The maximum amplitude of the group initially decays faster, but the decay rate appears to slow down. The large time asymptotic decay rate was predicted by the spatio-temporal theory to be proportional to \(t^{-1/2}\). The maximum amplitude depicted in Fig. 2 divided by \(t^{-1/2}\), i.e., \([F_{max}] / t^{-1/2}\) is plotted as a function of t in Fig. 3. It is seen that this ratio reaches practically a constant value at a dimensionless transient time of 0.01. For water flowing in a sheet of thickness 0.2cm with \(W_e = 1.002, U = 7.2\) m/sec. The dimensional transient time is therefore less than 2 x 10-4 sec. During this short transience, the decay rate is faster than \(t^{-1/2}\) as indicated by the large slope except between \(t = 0.001\) and 0.002. During this time period the slope of the curve changes sign, and thus the \([F_{max}]\) actually grows instead of decays momentarily. Fortunately this transient growth is short lived, and does not invalidate the stability predicted by the spatio-temporal theory.

**DISCUSSION**

The stability of a liquid sheet with respect to long varicose wave disturbances is analyzed on the basis of Taylor’s equation with three different linear theories. Both the pure temporal and pure spatial theories predict neutral oscillation without decay of disturbances in a liquid sheet. The spatio-temporal theory predicts a large time asymptotic decay of the disturbance. The predicted algebraic decay rate is confirmed by the solution of the initial value problem, after a short transient period. During this short transience, there is a short period in which the disturbance actually grows. However this short burst of growth decays quickly. The disturbance decays algebraically after the initial transience. The initial transience has not invalidated the spatio-temporal theory in this example. The initial growth may render the prediction of linear theory physically insignificant only when the transient growth is so large that the nonlinear effect becomes significant before the large time asymptotic behavior is approached. This situation is an exception rather than a rule. This is the reason linear theory has been so suc-
cessfully applied to the investigation of breakup of liquid sheets and jets [12]. However linear theory must be applied with care as expounded in this paper. The use of pure spatial theory should be avoided, since it is known to predict instability while the flow is actually stable [10]. Temporal theory is easier to use, but with some limitations. If it predicts instability, the flow may only be convectively unstable. Then the spatial growth of a disturbance may be inferred by invoking the Gaster theory [13] as was done by some workers [5,12,13]. However absolute instability in a liquid sheet escaped the prediction by pure temporal theory [9]. Here we have shown by comparing the prediction of spatio-temporal theory with the solution of the initial value problem that while both pure spatial and temporal theories predict incorrectly the large time sustained neutral oscillation, the spatio-temporal theory predicts correctly the large time algebraic decay of disturbances as they are convected downstream.

NOMENCLATURE

\( D \) dispersion function
\( F \) Free surface displacement
\( f \) Fourier transformation of \( F \)
\( h \) half sheet thickness
\( K \) dimensional wave number
\( k \) wave number
\( p \) pressure
\( S \) surface tension
\( t \) time
\( U \) Uniform velocity of basic flow
\( We \) Weber number
\( (x,y,z) \) Cartesian coordinates

Greek Symbols

\( \Omega \) dimensional frequency
\( \omega \) disturbance frequency
\( \delta \) Dirac delta
\( \rho \) density

Subscripts

\( i \) i-th spatial node

Superscripts

\( n \) n-th time step

REFERENCES

Figure 1. Amplification curves.

Figure 2. Time evolution of varicose waves in a liquid sheet, We=1.002.
Figure 3. Time history of disturbance growth, We=1.002.