On the Linear Stability of Compound Capillary Jets

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Abstract

Compound capillary jets are utilized in the manufacture of coated tablets within the pharmaceutical industry and present an interesting atomization problem. Differences in density and surface tension between the inner fluid (the medicine) and the outer fluid (the coating) provide for complex interactions relative to capillary instability. The present study was motivated by a ILASS 2006 presentation by Bian and Mashayek who developed a one dimensional nonlinear treatment of the instability. The present work was initially studied as a homework problem in a graduate class taught by the co-author at Purdue during the fall 2006 semester. An axisymmetric, inviscid linear instability analysis has been developed to compliment the Bian and Mashayek work and to provide insight into the droplet sizes formed under a variety of conditions. Inner and outer fluid density and surface tension are to be varied parametrically to assess their influence on the droplet sizes formed. The effect of the thickness of the outer fluid will also be assessed parametrically.

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Introduction.

A compound jet, see Fig.1, is comprised of two liquid jets, one of which, the inner, is surrounded by the other, the outer. This technique is used in the printing and pharmaceutical industries to obtain a fine ink jet or medicine wrapped in a capsule. The inner liquid is injected through a nozzle into a quiescent plenum containing the other liquid, shear forces at the periphery of the jet form a thin covering layer moving eventually with the same speed as the internal jet.

There are many publications concerned with the stability characteristics and breaking regimes of compound jets. The current study was motivated by the ILASS 2006 presentation by Bian and Mashayek [1], who developed a numerical model to simulate the behavior of 1-D viscous compound jet, and has been conducted as a homework computational project. The purpose of the study was to find a dispersion relation for an inviscid compound jet, and based on that, to investigate how such parameters as density, tension and radius of the outer liquid would affect the stability of the jet, given same parameters of the inner liquid remain unchanged. In general, the jet starts to generate droplets at the most unstable wave-numbers and highest growth rates. Based on these numbers the students had to make an estimation for the droplet parameters.

Another paper of interest is by Sanz and Meseguer [2], who also considered the stability of an inviscid compound jet. They used continuity and axial momentum as governing equations, involving velocities of inner and outer liquids in them. At linearization these velocities are eliminated to get the sought 4-th order dispersion relation. In our study we start with an assumption that the coordinate system is moving with the bulk speed of compound jet, and use continuity, kinematic and dynamic boundary conditions as governing equations. The velocities are represented by velocity potentials. After linearization a 2-nd order dispersion relation is obtained by solving the Bessel’s equation.

Assumptions

- Flow is axisymmetric, incompressible, inviscid, irrotational.
- Liquid columns are infinitely long, immiscible.
- No body forces present.
- Coordinate system is moving with the bulk speed of both jets, U, see Fig.1.
- Small-amplitude waves considered, velocity and liquid surface perturbations are small.
- Liquids are injected into the quiescent gas with the pressure equal to zero.
- Equal surface elevation for both liquids.

Governing equations.

Applying continuity to axisymmetric flow gives:

\[ \frac{\partial u_j}{\partial z} + \frac{1}{r} \frac{\partial}{\partial r} \left( r v_j \right) = 0 \]  
(1.1)

and considering that

\[ u_j = \frac{\partial \phi_j}{\partial z} \quad \text{and} \quad v_j = \frac{\partial \phi_j}{\partial r}, \]

continuity relations for two liquid jets are:

\[ \frac{\partial^2 \phi_j}{\partial z^2} + \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \phi_j}{\partial r} \right) = 0 \]  
(1.2)

The unsteady Bernoulli equation provides one of the two boundary conditions for the interfaces. Along the linearized interface radial position \( r = a_j \) this condition may be expressed:

\[ \frac{\partial \phi_j}{\partial t} + \frac{1}{2} \frac{\partial }{\partial r} \left( \rho_j \nabla \phi \cdot \nabla \phi_j - G_j \right) = 0, \]

(1.3)

where we neglect body forces \( G_j \) and higher order term \( \frac{1}{2} \nabla \phi_j \cdot \nabla \phi_j \), i.e. the dynamic pressure is a higher order quantity when one considers a coordinate system traveling with the bulk velocity of the jet.

The local pressure can be represented as a sum of the initial liquid pressure and a disturbance as:

\[ p_j = p^0_j + \hat{p}_j. \]

Considering that \( dp_j = d \left( p_j - p^0_j \right) \) we can rewrite eqs.(1.3) for each liquid jet in the form of

\[ \rho_j \frac{\partial \phi_j}{\partial t} = p^0_j - p_j. \]  
(1.4)

Consider surface about \( r = a_i \). Subtracting of eqs.(1.4) in this case yields

\[ \rho_2 \frac{\partial \phi_2}{\partial t} - \rho_1 \frac{\partial \phi_1}{\partial t} = \left( p^0_2 - p^0_1 \right) - \left( p_2 - p_1 \right), \]  
(1.5)

where the initial pressures can be written as:

\[ p^0_i = \frac{\sigma_i}{a_i} \quad \text{and} \quad p^0_2 = \frac{\sigma_2}{a_2}. \]  
(1.6)

The local pressures at the interface between two liquids are related by

\[ p_2 - p_1 = -\sigma_2 K_a. \]  
(1.7)

Here the axisymmetric surface curvature can be expressed as a sum of two terms,
\[ K'_n = \frac{1}{R_1'} + \frac{1}{R_2'} , \quad (1.8) \]

which are:
\[ \frac{1}{R_1'^{\text{eq}}} = \frac{1}{a_i + \eta} \left[ 1 + \left( \frac{\partial}{\partial z} (a_i + \eta) \right)^2 \right] \]

and
\[ \frac{1}{R_2'^{\text{eq}}} = \frac{1}{a_i + \eta} \left[ 1 + \left( \frac{\partial}{\partial z} (a_i + \eta) \right)^2 \right] . \]

Neglecting \( \frac{\partial^2 \eta}{\partial z^2} \) as a HOT term relative to unity and expanding \( \frac{1}{a_i + \eta} \) in Taylor’s series around \( \eta = 0 \) we rewrite (1.8) as
\[ K'_n = \frac{1 - \eta}{a_i} \frac{\partial^2 \eta}{\partial z^2} . \quad (1.9) \]

Combining eqs.(1.5) through (1.9) we obtain the linearized dynamic boundary condition for the interface:
\[ \rho \frac{\partial \phi_s}{\partial t} - \rho \frac{\partial \phi_p}{\partial t} + \frac{\sigma}{a_i} \eta + \frac{\sigma^2}{a_i} \frac{\partial^2 \eta}{\partial z^2} = 0 . \quad (1.10) \]

Consider surface about \( r = a_1 \). For this surface eqs.(1.4) can be represented as:
\[ \rho \frac{\partial \phi_s}{\partial t} = P_x^0 - p_x \quad \text{and} \quad \rho \frac{\partial \phi_p}{\partial t} = P_y^0 - p_y , \quad (1.11) \]

where we assume vacuum conditions \( \rho_x, \phi_x, P_x^0, p_x = 0 \) for this particular study. Here the pressures are related as:
\[ P_x^0 - P_y^0 = \frac{\sigma}{a_1} \quad \text{and} \quad p_x - p_y = -\sigma K'_n . \quad (1.12) \]

Following the derivation of eq.(1.9) we can similarly obtain
\[ K'_n = \frac{1 - \eta}{a_2} \frac{\partial^2 \eta}{\partial z^2} . \quad (1.13) \]

Combining eqs.(1.10) through (1.13) yields the dynamic boundary condition for the free surface
\[ -\rho \frac{\partial \phi_s}{\partial t} + \frac{\sigma}{a_2} \eta + \frac{\sigma^2}{a_2} \frac{\partial^2 \eta}{\partial z^2} = 0 . \quad (1.14) \]

Kinematic boundary conditions evaluated at \( r = a_1 \) lines can be written:
\[ \frac{\partial \phi_s}{\partial r} = \frac{\partial \eta}{\partial t} + \frac{\partial \phi_s}{\partial z} \frac{\partial \eta}{\partial z} . \quad (1.15) \]

Neglecting \( \frac{\partial \phi_s}{\partial z} \) as higher order terms, we can rewrite (1.15) for two radial mean levels as:
\[ \text{at } r = a_1, \quad \frac{\partial \phi_s}{\partial r} = \frac{\partial \phi_s}{\partial r} = \frac{\partial \eta}{\partial t} . \quad (1.16) \]

at \( r = a_2 \)
\[ \frac{\partial \phi_s}{\partial r} = \frac{\partial \eta}{\partial t} . \quad (1.17) \]

### Linearization of Governing Equations

We assume that both velocity potentials and surface elevation can be represented in the form of Fourier waves:
\[ \phi_j = F_{j}(r) \exp(\omega t + i k z) , \quad (2.1) \]
\[ \eta_j = \eta_{j0} \exp(\omega t + i k z) , \quad (2.2) \]

where \( F_{j}(r) \) is an unknown function and \( \eta_{j0} \) is some initial surface elevation.

Then plugging eq.(2.1) into (1.2) yields an expression for \( F_{j}(r) \):
\[ r^2 \frac{\partial^2 F_j(r)}{\partial r^2} + r \frac{\partial F_j(r)}{\partial r} - k^2 r^2 F_j(r) = 0 , \]

which is a modified Bessel equation.

We can write a solution for it for each liquid in the following form:
\[ F_j(r) = C^{(i)}_x (\nu) + C^{(i)}_y N_x (\nu) , \quad (2.3) \]
\[ F_x(r) = C^{(i)}_y (\nu) + C^{(i)}_x N_x (\nu) , \quad (2.4) \]

where \( C^{(i)}_x \) and \( C^{(i)}_y \) are constants to be defined. At the limit \( r \to 0 \) perturbations should vanish, which means that \( C^{(i)}_x = 0 \).

Consider liquid surface about \( r = a_1 \). Using eq.(2.1) we can write
\[ \phi_1 = F_1(a_1) \exp(\omega t + i k z) , \quad (2.5) \]
\[ \phi_2 = F_2(a_1) \exp(\omega t + i k z) , \quad (2.6) \]

where from eqs.(2.3) and (2.4) it follows that
\[ F_1(a_1) = C^{(i)}_x (\nu) , \]
\[ F_2(a_1) = C^{(ii)}_x (\nu) + C^{(i)}_x N_x (\nu) . \]

Plugging eqs.(2.5) and (2.6) into kinematic boundary condition (1.16) and using eq.(2.2) we obtain the first two relations to determine the unknown constants:
\[ C^{(i)}_x (\nu) = \frac{\eta_{j0}}{k} , \quad (2.7) \]
\[ C^{(ii)}_x (\nu) = \frac{\eta_{j0}}{k} . \quad (2.8) \]

Consider liquid surface about \( r = a_2 \). Here we similarly have
\[ \phi_2 = F_2(a_2) \exp(\omega t + i k z) , \quad (2.9) \]

where
\[ F_2(a_2) = C^{(i)}_x (\nu) + C^{(i)}_x N_x (\nu) . \]

Plugging eq.(2.9) into dynamic boundary condition (1.14) and using eq.(2.2) provides the third relation
\[ C^{(i)}_x (\nu) + C^{(i)}_x N_x (\nu) = \frac{\sigma \nu}{\rho \omega} \frac{1}{a_2^2} - k^2 \quad (2.10) \]

Solving eqs.(2.7), (2.8) and (2.10) for the unknown constants we obtain:
By this, the functions (2.3) and (2.4) and hence the equations for the velocity potentials at both liquid surfaces (2.1), are fully defined in terms of known variables.

**Dispersion relation.**

To get the dispersion relation for this study we use the dynamic boundary condition at the interface (1.10). Evaluating the velocity potentials at \( r = a_1 \) and after number of mathematical manipulations we can obtain

\[
\omega = \sqrt{\frac{k^2 \left( E \sigma_2 + \sigma_1 \right) - \left( \frac{E}{D} \sigma_2 + \sigma_1 \right)}{\rho_1 A - \frac{E}{D} \rho_2 B - \rho_2 B}},
\]

where the functions of \( k, A, B, D, E \), are defined as:

\[
A = \frac{I_n(k a_1)}{I_1(k a_1)},
\]

\[
B = \frac{I_n(k a_2)}{I_1(k a_1)},
\]

\[
D = K_1(k a_1) \frac{I_n(k a_2)}{I_1(k a_1)} + K_0(k a_2),
\]

\[
E = K_1(k a_1) \frac{I_n(k a_1)}{I_1(k a_1)} + K_0(k a_1).
\]

**Parametric study.**

In this study the influence of the several parameters of the outer liquid on the stability of the whole compound jet is observed. These parameters include density, surface tension and the mean radius \( a_2 \). The inner liquid is assumed to have the following parameters: \( \rho_1 = 1000 kg/m^3 \), \( \sigma_1 = 0.074 N/m \), \( a_1 = 5 mm \).

The variation of the density, see Fig.2, shows that as the outer liquid gets denser, the growth rate decreases by about 20%, while the range of disturbance wave-numbers does not change. This implies that the compound jet is more stable when the outer jet is denser. The reason could be that oscillation of molecules of the inner liquid close to the interface is damped by denser packed molecules of the outer liquid, thereby diminishing overall disturbances in the jet.

**Variation of density \( \rho_2 \) (kg/m\(^3\))**

\( a_2 = 7.5 mm, \sigma_2 = 0.074 N/m \)

\( \rho_2 = 250 \)

\( \rho_2 = 1000 \)

\( \rho_2 = 4000 \)

Figure 2. The effect of density.

The increase of the surface tension of the outer liquid, see Fig.3, results in the higher maximum growth rate with shorter range of unstable wave-numbers. That is the increase in the surface tension causes the amplification of low-frequency disturbances with damping of high-frequencies. In other words, the surface tension forces are weaker than the inertia forces of the long wave-length liquid particles. In contrast, when the liquid oscillates with higher frequency, the amplitude of motion of the particles is small, and the surface tension forces are able to lessen this amplitude to lower values.

**Variation of surface tension \( \sigma_2 \) (Nm)**

\( a_2 = 7.5 mm, \rho_2 = 1000 N/m^2 \)

\( \sigma_2 = 0.037 \)

\( \sigma_2 = 0.074 \)

\( \sigma_2 = 0.111 \)

Figure 3. The effect of surface tension.
The variation of the outer radius, see Fig.4, shows a clear trend towards enhanced stability as the radius of the outer liquid grows. This can be explained by the fact that the heavier the outer liquid gets, the harder it is for the inner liquid to disturb it according to conservation of momentum. We can also see that the range of the unstable wave-numbers widens towards the high-frequencies as the outer radius decreases. Which tells us, that the jet gets sensitive to even weak disturbances with small amplitudes.

**Droplet size approximate prediction.**

For each unstable wave-number on the stability curve there exists a corresponding droplet size, as a droplet can be potentially formed as a result of separation of a liquid segment from the bulk of the liquid. Here we consider only the droplets that can be generated at most unstable wave-numbers. We assume that at the moment of separation the droplets can be treated as liquid columns with the length equal to corresponding most unstable wave-length and the radius equal to $a_i$. After separation the droplets are assumed to form spherical bodies. Thus, for the approximate prediction of the droplet size we have

$$
\pi a_i^2 \lambda_{\text{max}} = \frac{4}{3} \pi r_d^3,
$$

where $\lambda_{\text{max}} = \frac{2\pi}{k_{\text{max}}}$. Then the corresponding droplet radius is

$$
r_d = \left( \frac{1.5\pi a_i^2}{k_{\text{max}}} \right)^{\frac{1}{3}}.
$$

We can rewrite eq.(2.11) factoring out $\rho_i$ and $\sigma_i$ as

$$
\omega = \sqrt{\frac{\pi \sigma_i^2 \lambda_{\text{max}}}{\rho_i}}
$$

That means that if the stability curve would have a maximum $\omega_{\text{max}}$, it would not depend on particular values of $\rho_i$ and $\sigma_i$, but on ratios of densities and tensions. At the same time, since both $a_i$ and $a_i$ are present in Bessel’s relations $A, B, D, E$, then $\omega_{\text{max}}$ will depend on each particular value of $a_i$ and radius ratio $\frac{a_2}{a_1}$. Noting this, we present the results for the droplet size predictions for the chosen inner radius $a_i = 5\text{mm}$.

**Figure 4.** The effect of outer radius.

**Figure 5.** Density variation with fixed radius ratio.

**Figure 6.** Tension variation with fixed radius ratio.
Here we consider several cases where one of the three parameters \( \left( \frac{a_2}{a_1}, \frac{\rho_2}{\rho_1}, \frac{\sigma_2}{\sigma_1} \right) \) has been varied at different values of the second parameter, while the third parameter has been held constant, totally six different combinations.

The plots of droplet sizes by varying density and surface tension at fixed radius ratio, see Figs. 5 and 6, confirm the trends observed on Figs. 2 and 3 where the increase in density and surface tension of outer liquid led to the lower values of the most unstable wave-numbers and hence to bigger droplet sizes.

The plots of varying radius and tension ratio at fixed density ratio on Figs. 7 and 8 repeat the stability behavior on Fig. 3 and 4. Here we can see again that the increase in surface tension and radius of the outer jet results in lower \( k_{\text{max}} \), and consequently in bigger droplet sizes.

The variation of radius and density ratio at fixed tension ratio, see Figs. 9 and 10 results in bigger droplets as radius and tension ratio are increased, as on Figs. 3 and 4.

Figure 7. Outer radius variation with fixed density ratio.

Figure 8. Surface tension variation with fixed density ratio.

Figure 9. Outer radius variation with fixed surface tension ratio.

Figure 10. Density variation with fixed surface tension ratio.

Overall, we can observe that with the set of parametric ratios considered here the droplet sizes lie in the range of 0.2..0.45 of the outer radius of compound jet. We can also note from Figs. 8 and 10 that when solely surface tension or density ratio is varied with a radius ratio below \( \frac{a_2}{a_1} = 1.5 \), the size of the droplets changes very little. This can be of benefit when both...
jet radii can be well controlled and the liquids of the compound jet are varied frequently, to produce droplets with stable sizes.

Conclusions.
This study is concerned with the stability of the compound jet with regard to variation of several parameters of the outer liquid while those of the inner liquid remain unchanged. The parameters varied were density $\rho_j$, surface tension $\sigma_j$, and the radius $a_j$.

The plots due to dispersion relation have shown some distinguishable trends which are the following. With denser outer liquid the compound jet is more stable, and the stability region does not change. If the outer liquid has a higher surface tension the lower wave-numbers are amplified while the higher ones are damped, and we can observe that the maximum growth rate is higher. In case of increased outer radius we see that growth-rate, as well as the range of unstable wave-numbers, decrease.

The stability characteristics obtained provide us with a tool for an approximate prediction of droplet sizes, as droplets are pinched off from the bulk of the liquid. The plots of droplet sizes with continuous variation of radius, density and surface tension ratio show an agreement with trends observed previously where these parameters were chosen discretely.

Nomenclature.

<table>
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<tr>
<th>Symbol</th>
<th>Description</th>
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<tbody>
<tr>
<td>$a$</td>
<td>radius of an undisturbed liquid surface</td>
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<tr>
<td>$d$</td>
<td>droplet radius</td>
</tr>
<tr>
<td>$I_0$</td>
<td>Bessel function of the 1st kind of order 0</td>
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<td>$I_1$</td>
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<tr>
<td>$K_1$</td>
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<td>$K^*$</td>
<td>axisymmetric surface curvature</td>
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<td>$k$</td>
<td>wave number</td>
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<td>$p$</td>
<td>local pressure</td>
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<td>$\dot{p}$</td>
<td>pressure perturbation</td>
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<td>pressure before any disturbance</td>
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<td>$r_d$</td>
<td>droplet radius from the inner liquid</td>
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<td>$r, z$</td>
<td>radial and axial coordinates</td>
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<td>$\phi$</td>
<td>velocity perturbation potential</td>
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Subscripts.

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<td>$g$</td>
<td>ambient gas</td>
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<td>$j$</td>
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<tr>
<td>$in$</td>
<td>interface between two liquids</td>
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<tr>
<td>$out$</td>
<td>outer surface</td>
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<td>$\text{max}$</td>
<td>most destabilizing disturbance</td>
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Superscripts

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References.