Near nozzle analysis of a liquid filament under Rayleigh breakup conditions created from laminar rotary spraying of oil-in-water emulsions

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Abstract
A liquid jet resulting from the laminar rotary spraying of an oil-in-water emulsion (O/W) with a viscosity on the order of 50 mPa s and a surface tension on the order 50 mN/m has been studied by means of a high-speed camera. The liquid flow rate and rotational speed of the rotary sprayer are tuned so that the liquid filament breaks up into droplets under Rayleigh breakup conditions. From the high-speed imagery we consider in detail the dominant forces, which define the shape of the liquid jet near the nozzle exit. We use the formation of Rayleigh disturbances as a tracing mechanism across multiple high-speed video frames to determine the role that rotational forces, surface tension, viscous forces and wind resistance play on the shape of the liquid filament as well as the formation of resulting droplets. From this analysis it is determined that rotational forces play the dominant role, thus resulting in a simplified parametric model of the liquid jet trajectory based on the Rossby number only. This model is compared to a previous model defined in the Frenet–Serret frame of reference and shown to be the same under simplifying assumptions.

Keywords: Rayleigh breakup; Rotary spray; Rossby number; High-speed camera.

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Introduction

The difficulty of spray processing emulsions arises from their sensitivity to fluid mechanical stress, which under standard spray conditions causes the emulsive structure to break down. This in turn causes loss of encapsulation of functional components that drive the need for the emulsive structures in the first place.

Rotary spraying of liquids under laminar conditions is one method for handling shear sensitive materials. This spray regime is a jetting regime characterized by low Reynolds and Weber numbers, in which the liquid jet breaks up under Rayleigh breakup conditions [1]. By maintaining low Reynolds and Weber numbers, the shear stress remains low, thus reducing loss of encapsulation due to nozzle and spraying effects. Furthermore, the coupling of rotary spraying with Rayleigh breakup of the liquid jet produces monodisperse droplets with the added benefit that the droplets are smaller than the nozzle diameter [2].

A common approach for analyzing the Rayleigh breakup behavior of a liquid filament into droplets is to approximate the liquid filament as an axisymmetric jet, since the curvature of the filament arc is large in comparison to the radius of the filament [3]. The axisymmetric jet is then analyzed according to Rayleigh breakup theory [4, 5]. This approach has been employed by Gramlich and Piesche [6] and Părău et al. [7] to analyze Rayleigh breakup behavior of liquid filaments formed by rotary spraying under laminar conditions.

Approximating the liquid filament as an axisymmetric jet requires a set of equations defining the trajectory of the liquid filament coupled with conservation of mass and momentum. In particular, the fluid velocity, pressure and filament radius are defined as a function of material properties, boundary conditions and position. In the analysis of Gramlich and Piesche [6] and Părău et al. [7], the formulation is set up in a Lagrangian framework, which after simplifying results in a system of equations in a Frenet-Serret frame of reference containing a tangential component, a normal component and a binormal component, the latter of which may be zero.

This article addresses only the shape and trajectory of the liquid filament, which is jetting from rotary sprayer under laminar conditions. Rather than coupling the conservation equations directly into the model of the fluid trajectory, a force balance is applied to an arbitrary fluid element in a rotating frame of reference. The resulting equations of motion, when translated to an inertial frame of reference, result in a Cartesian formulation of position and velocity for all points along the trajectory.

Mathematical Model

A liquid filament with an initial nozzle speed \(w_0\) exits the nozzle of a rotary sprayer with radius \(r_0\) and constant angular speed \(\omega\) as shown in Figure 1. Within a rotating frame of reference with origin \(O\), centrifugal force and Coriolis force accelerate the segment of the filament. Neglecting gravity and wind resistance and assuming the liquid filament behaves as a string, the viscous and surface tension forces act primarily in the direction of the segment and are grouped together as the tension vector, \(\mathbf{T}\). The resulting trajectory of the filament segment is defined as a function of time, \(t\), by the position vector \(\mathbf{r}(t) = (x(t), y(t))\) with arc length \(s(t)\) and velocity \(\mathbf{w}(t) = (u(t), v(t))\). Applying the force balance at an arbitrary point, \(P\), on the trajectory determines the acceleration \(\mathbf{w}'(t)\) of a segment with length \(\delta s\) and per unit length density, \(\rho\), as follows:

\[
\rho \delta s \mathbf{w}'(t) = -\rho \delta s \mathbf{\omega} \times (\mathbf{\omega} \times \mathbf{r}(t)) - 2\rho \delta s \mathbf{\omega} \times \mathbf{w}(t) + \mathbf{T}. \tag{1}
\]

The magnitude of the tension depends on position along the trajectory, which is inherently defined in terms of time. Furthermore, the tension force vector is parallel with but acting opposite to the velocity. Therefore rewriting (1) in component form, dividing by segment mass, and writing in dimensionless terms results in the parametric form as follows:

\[
\begin{align*}
\dot{X}(T) &= \frac{X(T)}{Rb^2} + 2\frac{Y(T)}{Rb} - F(T) \dot{X}(T) \\
\dot{Y}(T) &= \frac{Y(T)}{Rb^2} - 2\frac{X(T)}{Rb} - F(T) \dot{Y}(T), \tag{2}
\end{align*}
\]

where \(F(T)\) represents a scaling of the tension.
forces. The equation uses the scaling:

\[ X = \frac{x}{r_0}, \quad Y = \frac{y}{r_0}, \quad T = t \frac{\omega_0}{r_0}, \quad Rb = \frac{w_0}{r_0\omega}, \]

where the characteristic number, \( Rb \), is the Rossby number and the overhead dot represents the derivative \( \frac{d}{dt} \). The dimensionless terms for the arc length, position vector and velocity vector follow from the scaling, which results in:

\[ S(T) = \frac{s(t)}{r_0}, \quad R(T) = \frac{r(t)}{r_0} = (X(T), Y(T)), \quad W(T) = \frac{w(t)}{w_0} = (U(T), V(T)), \]

where \( U(T) \) and \( V(T) \) are the horizontal and vertical velocity components, respectively.

Equation (2) is an initial value problem with \( R(0) = (\cos \theta, \sin \theta) \) and \( W(0) = (\cos \theta, \sin \theta) \), where the phase angle, \( \theta \), represents the angular position of the nozzle exit. After relaxing the tension term, the initial value problem has the closed form solution:

\[ X(T) = T \frac{\alpha}{Rb} \sin \left( \frac{T \alpha}{Rb} - \theta \right) + (T + 1) \cos \left( \frac{T \alpha}{Rb} - \theta \right), \]
\[ Y(T) = T \frac{\alpha}{Rb} \cos \left( \frac{T \alpha}{Rb} - \theta \right) - (T + 1) \sin \left( \frac{T \alpha}{Rb} - \theta \right), \]

which defines the trajectory of a filament segment in a rotating frame of reference.

In Equation (3) the tension term has been relaxed, which implies that the arbitrary filament segment is not bound to adjacent segments as required by a tension encompassing model. Therefore the trajectory defines the equation of motion of a particle exiting the nozzle at time \( T = 0 \). Furthermore, the particle trajectory in a rotating frame of reference is equivalent to the liquid filament shape in an inertial frame of reference.

**Comparison to a Frenet-Serret frame of reference**

Granich and Piesche [6] develop the differential equations defining the liquid jet trajectory in a Frenet-Serret frame of reference resulting in the tangential equation and the normal equation. In these equations we write the rotational number, \( Ro = \frac{1}{\omega_0 \alpha_0} \), in terms of the Rossby number, where \( \alpha_0 \) is the initial filament radius. Neglecting viscous forces, surface tension forces and wind resistance. The tangential equation simplifies to

\[ W \frac{dW}{dS} - \frac{R}{Rb^2} \frac{dR}{dS} = 0, \]

which after applying the boundary value, \( (S = 0, R = 1, W = 1) \), further reduces to

\[ W^2 - \frac{R^2}{Rb^2} = 1 - \frac{1}{Rb^2}. \]

Likewise the normal equation reduces to

\[ W^2 \kappa + \frac{R}{Rb^2} \cos \alpha - 2W = 0, \]

where \( \kappa \) is the mean curvature and \( \alpha \) is the angle of inclination, as described in [6].

The results of the inertial trajectory formulation (3) are substituted into the above tangential and normal equations to show that they are equivalent solutions. After substituting, the tangential equation becomes

\[ W^2 - \frac{R^2}{Rb^2} = \left( \frac{\alpha}{Rb} \right)^2 + \frac{(X^2 + Y^2)}{Rb^2} - \left( \frac{\alpha}{Rb} \right)^2 - \left( \frac{(X^2 + Y^2)}{Rb^2} \right)^2 \]
\[ = 1 - \frac{1}{Rb^2}. \]

The normal equation requires that we first resolve \( \kappa \) and \( \cos \alpha \), which after simplification yield:

\[ W^3 \kappa = |\ddot{X}Y - \dot{Y} \dot{X}|, \quad W R \cos \alpha = |\ddot{X}Y - \dot{Y} \dot{X}|. \]

Multiplying (5) by \( Rb W \) and substituting yields:

\[ RbW^3 \kappa + \frac{WR}{Rb} \cos \alpha - 2W^2 \]
\[ = Rb \left| \ddot{X}Y - \dot{Y} \dot{X} \right| + \frac{1}{Rb} \left| \ddot{X}Y - \dot{Y} \dot{X} \right| - 2W^2 \]
\[ = \left( \frac{2}{Rb} + \frac{T^2}{Rb^2} + \frac{T^2}{Rb^2} \right) + \left( \frac{2}{Rb^2} + \frac{T^2}{Rb^2} + \frac{T^2}{Rb^2} \right) \]
\[ - 2 \left( \frac{2}{Rb} + \frac{T^2}{Rb^2} + \frac{T^2}{Rb^2} \right) \]
\[ \equiv 0. \]

**Experimental details**

The measurement test-stand was set up according to Figure 2. Further details of the experiment setup including preparation of the O/W emulsion can be found in Dubey [8]. The zero shear viscosity of the fluid, \( \eta_s \), was estimated as 0.06 Pa s and the surface tension \( \gamma_s \), was estimated as 0.04 N/m. The experiment conditions are summarized in Table 1.

Two spray nozzles were mounted opposite to each other on the rotor, each with orifice diameter of 0.3 mm. The diameter of the rotor measured as the distance from nozzle exit to opposite nozzle exit was 68.55 mm, thus the corresponding radius, \( r_0 \), is 34.28 mm.
Figure 2. Schematic diagram of pressure assisted rotary spray measurement.

<table>
<thead>
<tr>
<th>Experiment parameters</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rotor diameter to nozzle exit</td>
<td>68.55 mm</td>
</tr>
<tr>
<td>Nozzle orifice diameter</td>
<td>0.3 mm</td>
</tr>
<tr>
<td>Pixel size</td>
<td>56.2 µm</td>
</tr>
<tr>
<td>Camera frame rate</td>
<td>5000 fr/s</td>
</tr>
<tr>
<td>Rotational speed</td>
<td>1026 RPM</td>
</tr>
<tr>
<td>Dynamic viscosity</td>
<td>0.06 Pa s</td>
</tr>
<tr>
<td>Surface tension</td>
<td>0.04 mN/m</td>
</tr>
<tr>
<td>Density</td>
<td>980 kg/m³</td>
</tr>
<tr>
<td>Nozzle exit velocity</td>
<td>2.5 m/s</td>
</tr>
<tr>
<td>Feed pressure</td>
<td>2 bar</td>
</tr>
</tbody>
</table>

Table 1. Summary of experiment conditions.

Before operation the high-speed camera, Memrecam fx RX-6, was set to measure at 5000 frames per second (fr/s), which results in an image size of 512 pixels wide × 500 pixels high. The camera focus was adjusted to the nozzle exit and a scale was placed to determine the size of an individual pixel to be 56.2 µm in resulting video frames.

During operation the rotational speed of the motor was adjusted to approximately 1000 RPM and the actual rotational speed was measured using the resulting data from the high-speed camera. Aligning the nozzle image after three full revolutions of the rotary sprayer with the nozzle position at the start of the first revolution resulted in a total time of 175.4 ms and therefore a measured rotational speed of 1026 RPM. It follows that the angular velocity of the nozzle tip, \( \omega_r \), is 3.68 m/s.

The pump was adjusted, such that a pressure of 2 bar was maintained at the inlet of the rotary sprayer while rotating. The average flow rate of 21 mL/min was calculated over time by measuring the height change and thus volume change of the bulk, which corresponds to a mean fluid velocity of 2.5 m/s passing through the orifice nozzle.

Results and Discussion

An image composed by superimposing 38 consecutive frames of video capture of the counterclockwise rotating nozzle is shown on the left in Figure 3. The video capture is ordered such that the rightmost liquid jet represents the liquid jet captured in the first video frame and each adjacent liquid jet corresponds to the next video frame.

In observing the composite image, the development of Rayleigh disturbances and eventual Rayleigh breakup are highly visible. Rayleigh disturbance can be recognized by an optical trough traveling from lower right to upper left in the image, where the optical trough is the result of a thinning of the light-gray liquid filament allowing the dark background to become more visually dominant.

The composite image enables the reader to visually trace the Rayleigh disturbance backward to the start of their formation, as shown by the white lines in the right image of Figure 3. Tracing several disturbances back to their formation leads to three important observations: (i) the Rayleigh disturbances are propagating in a line across the composite image, (ii) the formation of Rayleigh disturbances can be recognized a short distance after nozzle exit, and (iii) the lines formed by tracing Rayleigh disturbances fan out when tracing from right to left.

According to Keller [9], Rayleigh disturbances travel with the fluid at the same speed of the fluid. Hence they provide a tracking mechanism for measuring fluid velocity. Consequently, a linearly propagating Rayleigh disturbance implies that the disturbance in addition to the fluid surrounding the disturbance are moving at a constant velocity and are not accelerating. It follows from observation (i) that the fluid surrounding the Rayleigh disturbances experiences zero acceleration, that is, it can be assumed that net forces surrounding the Rayleigh disturbances are zero.

The lines in the right image of Figure 3 begin in the lower right at visible start of the Rayleigh disturbance\(^1\) and follow the disturbances as they progress to the upper left. A quantitative measurement to observation (ii) estimates the first Rayleigh disturbance appearing at 2.5 mm from the nozzle exit. It is therefore reasonable to set an upper bound on distance from the nozzle, above which, the net forces surrounding a Rayleigh disturbance are zero.

The fanning behavior observed when tracing the disturbances from right to left further demonstrates that the liquid filament is acting as a sequence of

\(^1\)The visible start of the Rayleigh disturbances were taken from the unscaled composite 512 pixel × 500 pixel image, which may not reflect what is visible in the displayed figure.
distinct fluid particles with each particle traveling on its own constant velocity path. The speed and direction of the particle are fixed once the net force acting on the particle is effectively zero.

The spacing between two adjacent liquid jets in Figure 3 remains constant along the length of the jets and across all jets. This implies that the fluid particles are not accelerating in a direction normal to the filament and additionally the magnitude of the normal velocity is constant across all particles.

In the tangential direction of the liquid filament local acceleration of fluid particles can be observed. In particular where the Rayleigh disturbances grow and lead to detachment. At this point the surface tension forces dominate locally and cause the ends of the detached filament to pull together and form a droplet. However, based on observation (i), the local behavior is not significant enough to effect the shape of the liquid jet as a whole.

**Velocity components**

To accurately compare the liquid filament trajectory defined by (3) to the image results requires a more accurate value than the estimated mean fluid velocity, \( \bar{w}_0 \), of 2.5 m/s exiting radially from the nozzle. It follows from the above analysis that the liquid jet is experiencing minimal acceleration. Thus, a more accurate determination of fluid velocity can be calculated from the composite image by following one of the disturbance paths. Since the velocity of the disturbance is the vector sum of the nozzle exit velocity, \( w_0 \), and the angular velocity at the nozzle tip, \( \omega r_0 \), the Rossby number can be calculated directly as:

\[
R_b = \sqrt{\frac{w_{\text{disturb}}^2}{\omega^2 r_0^2}} - 1, \tag{6}
\]

where \( w_{\text{disturb}} \) is the speed of the propagating disturbance. Selecting the two points, displayed highlighted with a black dot over a white dot on Figure 3, the Rayleigh disturbance travels 538 pixels in 33 frames or 4.6 m/s. Applying this velocity and the angular velocity of 3.68 m/s to (6) yields a Rossby number, \( R_b \), of 0.74. It follows that a better estimate of the nozzle exit velocity, \( w_0 \), is 2.72 m/s. These values are summarized in Table 2 along with other resulting values.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nozzle exit velocity</td>
<td>2.72 m/s</td>
</tr>
<tr>
<td>Rotor x-coordinate</td>
<td>641 px</td>
</tr>
<tr>
<td>Rotor y-coordinate</td>
<td>-556 px</td>
</tr>
<tr>
<td>Initial phase angle</td>
<td>107°</td>
</tr>
<tr>
<td>Phase angle rate</td>
<td>1.23°/fr</td>
</tr>
<tr>
<td>Rossby number</td>
<td>0.74</td>
</tr>
<tr>
<td>Reynolds number</td>
<td>7</td>
</tr>
<tr>
<td>Gas Weber number</td>
<td>27</td>
</tr>
</tbody>
</table>

**Table 2. Calculated values and characteristic numbers.**

The Reynolds number, \( \text{Re} = \frac{\rho w_0 r_0}{\eta} \), and gas Weber number, \( \text{We} = \frac{\rho w_0^3 r_0}{\gamma} \), are calculated from the
resulting nozzle exit velocity, resulting in a value of 7 and 27, respectively.

Filament trajectories

The filament trajectories were calculated using (3) and scaled to pixel coordinates. The phase angle, $\theta$, was based on the nozzle angular position, $\theta_0$, in the first image frame and adjusted according to $\theta(fr) = \theta_0 + \frac{\omega}{frate}(fr - 1)$, where $fr$ is the frame number and $frate$ is the frame rate of 5000 frames/second. Together with $\theta_0$, the center of the rotor in image coordinates, $O = (\text{width, height})$ were optimized to ensure position and direction of the liquid filament were correct at the start of the visible part of the filament in each image frame.

This optimization resulted in a starting angular position of $\theta_0 = 107^\circ$ and a center of rotation of (641, -556) pixels, where (1, 1) pixels represents the lower left pixel in the image.

The resulting filament trajectories\(^2\) were overlaid on the composite image as shown on the left Figure 4. Good alignment has been achieved in the bottom half of the image. This region represents the liquid filament closest to nozzle exit. Farthest from the nozzle at the top of the image as magnified in the right image of Figure 4, the trajectory alignment deviates by approximately 1 mm from the measured filament. The slight deviation of the trajectory from the measurement can be explained by wind resistance playing a role, since wind resistance will tend to force the trajectory to the right in this image.

The earlier claim based on the analysis of Rayleigh disturbances stated that net forces surrounding a Rayleigh disturbance were zero. However, the strong agreement between the measured trajectories and calculated filament trajectories, which were based entirely on the Rossby number, allows the earlier claim to be relaxed. The trajectory was calculated in a rotating frame of reference applying only Coriolis and centrifugal forces, which translate to inertia in an inertial frame of reference. Therefore the shape of the liquid filament is determined by inertia, that is until wind resistance becomes significant. Moreover, the Rayleigh disturbances have already developed by the time wind resistance affects the fluid trajectory.

Summary and outlook

Using a composite image created from high-speed video capture of a liquid filament jetting from a rotary sprayer under laminar conditions with a Reynolds number of 7 and gas Weber number of 27, we were able to demonstrate that Rayleigh disturbances can act as a tracer in determining fluid velocity within the filament. Since the disturbances were moving in straight lines across the image, we could conclude that inertia dominates the filament’s shape. Moreover, though important to Rayleigh breakup of a liquid filament, the viscosity, surface tension and wind resistance are significantly less important in determining the shape of the liquid filament.

A mathematical model was developed in a rotating frame of reference using the balance of Coriolis, centrifugal and tension forces. When viscosity, surface tension and wind resistance are neglected, the shape of the liquid jet is based entirely on the

\(^2\)For clarity purposes, only seven trajectories were displayed, however the fitting characteristics of the omitted trajectories are in agreement with the represented trajectories.
Rossby characteristic number. Applying this model to the composite image showed a close match, which further supports that viscosity, surface tension and wind resistance can be neglected from the shape of the liquid filament under the conditions of this experiment.

The presented model greatly simplifies the shape of the liquid filament jetting from a rotary sprayer, however, it applies only when viscous forces, surface tension and wind resistance can be neglected. It remains to determine the exact conditions, under which, these forces can be neglected. A proposed approach would be to compare with the trajectories resulting from the equations of Gramlich [6] and Părău [7].

References


