Spatial Stability Analysis of a High Speed Liquid Jet

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Abstract
A spatial linear stability analysis of a liquid jet with a locally parallel base flow is performed. It is found that a pocket of absolute instability is present in the region closest to the jet exit, and that the most unstable mode occurs at the transition location between the absolutely and convectively unstable regions of the flow. Comparison with experimental results had previously employed the assumption that the wave velocity is close to the jet velocity, but the eigenvalues obtained suggest a much smaller wave velocity. This discrepancy is resolved by accounting for velocity changes at the jet surface.

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Introduction

Combustion systems, agricultural sprays, diesel and automotive engines, etc., utilize injectors or atomizers in order to deliver liquids in droplet form to their desired target. The study of atomization of liquid jets comprises the evolution in time and space of perturbations in the jet. The mechanism for the generation of these perturbations, however, is not well understood. These perturbations grow at a rate prescribed by the jet’s characteristic modes, until they eventually evolve and pinch off as droplets. Typically, the evolution of these perturbations in a given flow is studied via computational fluid dynamics (CFD) which, of course, requires initial and boundary conditions. These conditions are provided by either experimental measurements, correlations based on these measurements or simple theoretical models. These correlations, however, do not explain the physical mechanism involved and, for high speed jets, the models used to date either do not match experimental observations or are not self consistent. This paper provides an analytic model describing the temporal and spatial evolution of perturbations in a liquid jet; and, in doing so, it also explains the underlying mechanism responsible for the generation of perturbations in this liquid jet.

The subject of stability analysis in liquid jets has been studied for over a century. It began with Lord Rayleigh’s 1878 linear analysis of an inviscid low speed jet with no gas [1]. Weber [2] expanded Rayleigh’s work by including effects of viscosity and it was not until 1975 that aerodynamic effects were included by Sterling and Sleicher [3]. Two major limitations of these analyses is the assumption of an infinite liquid column and the lack of analysis of the spatial evolution of the perturbations. Brennen [4] analyzed the spatial evolution of these perturbations and relaxed the infinite liquid jet column assumption; he based his analysis on the development of a boundary layer inside the injector, compared his results with Hoyt and Taylor’s [5] liquid jet experiments, and obtained good agreement provided the wave velocity is close to the jet’s velocity. Heister and Yoon [6] categorize and compare these linear stability theories. A more detailed analysis of Brennen’s work shows that the wave velocity is smaller than the jet’s velocity, and thus, the calculated eigenvalues do not match the experimental results. This apparent discrepancy is the motivation for the current study.

Huerre [7] introduced the concept of absolute and convective instabilities applicable to parallel base flows. Koch, [8] then proposed a criteria to find the most unstable modes of flows containing regions with both types of instabilities. This work utilizes these concepts to explain the mechanism responsible for the generation of perturbations in a liquid jet. Finally, the results are shown to be in agreement with Hoyt and Taylor’s experiment.

Problem Formulation

It is assumed that a two dimensional, viscous, laminar, Blasius boundary layer develops inside the injector. As the flow exits the nozzle \((x = 0)\), the boundary condition at \(y = 0\) relaxes from a no slip to a free surface. This relaxation causes the appearance of an inflection point in the velocity profile which, according to Rayleigh’s inflection point criterion [9], is a necessary (but not sufficient) condition for inviscid instability. A schematic of this flow is shown in Figure 1.

Figure 1. Schematic of flow at the nozzle exit.

The non-dimensional variables are defined as follows:

\[
x_j = \frac{x_j^*}{\delta_2^*}, \quad u_j = \frac{u_j^*}{U_j^*}, \quad t = \frac{U_j^* t^*}{\delta_2^*},
\]

\[
p = \frac{p^*}{\rho^* U_j^* U_2^*}, \quad \omega = 2\pi f^* \frac{\delta_2^*}{U_j^*}. \quad (2)
\]

Stared \((\quad)^*\) variables are dimensional quantities; \(\delta_2^*\) corresponds to the momentum thickness at the jet exit, and it is assumed constant downstream of the exit; \(u_j^*\) is the velocity vector containing all velocity components, \(u, v\), in the streamwise and normal direction respectively; \(t\) is the nondimensional time; \(p\) is the pressure; \(\omega\) the angular frequency, \(f^*\) frequency in \(Hz\); and \(U_j^*\) is the streamwise velocity at the jet centerline. Two nondimensional parameters arise, the momentum thickness-based Reynolds and Weber numbers,

\[
Re_{\delta_2^*} = \frac{\rho^* U_j^* \delta_2^*}{\mu^*}, \quad We_{\delta_2^*} = \frac{\sigma^*}{\rho^* \delta_2^* U_j^*}, \quad (3)
\]

where \(\sigma^*\) represents the surface tension and \(\mu^*\) the liquid viscosity.

The edge velocity, \(U_e\), (velocity at \(y = 0\)) is zero at separation and it is assumed to accelerate as
it moves downstream according to Goldstein’s [10] analysis of a wake behind a flat plate. The resulting profile is plotted against streamwise location, $x^* / L^*$, in Figure 2, where $L^*$ is the injector length.

![Figure 2. Velocity at free surface (edge velocity). The circle, o, corresponds to the computed velocity where experimental measurements from Hoyt and Taylor [5] are reported.](image)

Finally, it is assumed that the gas exerts negligible stress on the liquid so that the conditions at the interface are constant normal and zero shear stress.

**Base Flow**

A base flow proposed by Brennen [4] where the mean velocity distribution is approximated by a Gaussian is used in the following analysis. This flow is defined as

$$w = \frac{1 - U}{1 - U_c} = e^{-ln(2)(y/b)^2}$$

(4)

where $w = 1 - U$ is the deficit velocity, $w_c = 1 - U_c(x) = 1 - U(x,0)$ corresponds to the deficit velocity at the free surface $y = 0$, and $b(x)$ is the half width of the boundary layer ($w(x,b) = \frac{1}{2}w_c(x)$) and is obtained from the definition of momentum thickness

$$\frac{1}{b} = \sqrt{\frac{\pi}{4ln2}} \left[ w_c - \frac{w_c^2}{\sqrt{2}} \right].$$

(5)

The edge velocity or $U_c$ is obtained from Goldstein’s [10] analysis of a two dimensional wake behind a flat plate (Figure 2). Figure 3 shows the Gaussian base flow velocity profile. Note that this base flow satisfies the zero shear stress condition $\partial_y U = 0$ at the free surface boundary $y = 0$ as stated earlier.

![Figure 3. Base flow velocity profile.](image)

**Governing Equations**

The velocities and pressure are expressed as a sum of mean flow and perturbation waves such that

$$[u_j, p_j](x, y, t) = [U_j, P_j](x, y) + [\hat{u}_j, \hat{p}_j](y)e^{i\alpha(x-ct)}$$

(6)

where $\alpha$ represents the wavenumber, $c$ the phase velocity, $\omega = \alpha c$ the frequency ($\alpha$, $c$, and $\omega$ are complex) and derivatives are given by

$$\partial_t u_j = -(i\alpha)\hat{u}_j e^{i\alpha(x-ct)}$$

$$\partial_x u_j = \partial_x U_j + (i\alpha)\hat{u}_j e^{i\alpha(x-ct)}$$

(7)

$$\partial_y u_j = \partial_y U_j + \partial_y \hat{u}_j e^{i\alpha(x-ct)}.$$

Viscous effects are accounted for in the base flow solution and, since viscosity is a stabilizing mechanism for free shear flows, an inviscid formulation for the perturbation analysis is justified. It is important to mention that viscosity is a destabilizing mechanism in wall bounded flows. Thus, introducing these wave forms into the inviscid Euler equations, linearizing them and assuming a parallel base flow ($V \ll U$ and $\partial U / \partial x \ll 1$), the equations governing the perturbations are found:

$$i\alpha\hat{u} + \partial_y \hat{\hat{u}} = 0$$

(8)

$$i\alpha\hat{u}(U - c) + \frac{\partial U}{\partial y} + \alpha \hat{p} = 0$$

(9)

$$i\alpha\hat{u}(U - c) + \frac{\partial \hat{p}}{\partial y} = 0.$$  

(10)

Non-axisymmetric effects are neglected based on Squire’s theorem [11] which states that, for parallel flows, the most unstable two-dimensional mode will be larger than any three-dimensional mode. Curvature effects are also ignored based on this theorem.

This set of equations, along with the following...
change of variables [4]  
\[ z = y/b, \quad k = (1 - c)/w_c, \quad \eta = ab, \]  
reduce to a single equation for the normal perturbation velocity, \( \hat{v}(y) \). This equation can be thought of as a global form of Rayleigh’s equation 
\[ \left( \frac{w}{w_c} - k \right) \left( \frac{d^2 \hat{v}}{dz^2} - \eta^2 \hat{v} \right) - \hat{v} \frac{d^2 (w/w_c)}{dz^2} = 0. \]  
(12)

When this ordinary differential equation (ODE) is supplemented with two homogeneous boundary conditions at \( y = 0 \) and \( y \to \infty \) (given below), a two point boundary value problem (BVP) is obtained with \( k \) and \( \eta \) as eigenvalues and \( \hat{v} \) as unknown. The global eigenvalues \( k \) and \( \eta \) resulting from a given solution of Rayleigh’s equation (12) are independent of \( z \), since \( w/w_c \) and its second derivative in this formulation are also independent of \( x \). The local eigenvalues, \( \alpha \) and \( c \) can then be obtained for each streamwise location using (11).

**Boundary Conditions**

Assuming that the interface (\( y = 0 \)) has a wave-like form similar to equation (6), with the exception that its amplitude does not depend on the normal coordinate, \( y \), then 
\[ h(x, t) = \bar{h}e^{i\alpha(x-ct)} \]  
(13)

where \( h \) represents the amplitude of a surface wave. With this, the linearized kinematic and dynamic boundary conditions at the surface respectively become 
\[ \hat{v}(0) = i\alpha \bar{h}[U_c - c]. \]  
(14) and 
\[ \hat{\rho}(0) = \frac{1}{We e_{\delta_2}} \alpha^2 \bar{h}(0). \]  
(15)

Where the Weber number based on momentum thickness, equation (3), is used.

Combining the above boundary conditions with the momentum and continuity equations and applying the change of variables (11) yields the first boundary condition 
\[ \frac{1}{\hat{v}(0)} \frac{\partial \hat{v}}{\partial z} = \frac{-1/b}{We e_{\delta_2}} \left( \frac{\eta}{w_c(1-k)} \right)^2 \]  
(16)

It is convenient to normalize \( \hat{v} \) such that \( \hat{v}(0) = 1 \) and, since surface tension is a stabilizing mechanism [4], it is not included in this analysis (neglecting its effects corresponds to an infinite Weber number). Therefore, 16 reduces to a Neumann boundary condition, 
\[ \frac{\partial \hat{v}}{\partial z} = 0. \]  
(17)

The far field boundary condition follows from an asymptotic analysis of Rayleigh’s equation (12). In the limit when \( z \to \infty, w/w_c \ll 1 \). Then, the requirement that the disturbances decay as \( z \to \infty \) gives 
\[ \lim_{z \to \infty} \left[ \frac{\partial \hat{v}}{\partial z} + \eta \hat{v} \right] = 0. \]  
(18)

The stability analysis has been reduced to solving Rayleigh’s equation (12) for \( \hat{v} \) with eigenvalues \( \eta \) and \( k \) subject to boundary conditions (17) and (18). This BVP is solved via a shooting method using the standard fourth order Runge-Kutta scheme.

**Spatial Analysis**

In a spatial stability analysis, the wavenumber \( \alpha \) in equation (6) is complex and the frequency \( \omega = \alpha c \) is real. Therefore, spatial stability theory considers disturbances that grow in space only. In this framework, a wave is decomposed as 
\[ e^{i\alpha(x-ct)} = e^{-\alpha_i x}e^{i\alpha_i x}e^{-i\omega_i t}. \]  
Here \( \alpha_i \) represents the spatial growth rate, for when \( \alpha_i < 0 \), the wave will grow unbounded in space.

If a perturbation is introduced in a given flow, it will either

1. Be convected downstream by the mean flow, or

2. Travel both upstream and downstream and affect the entire flow

as it is being amplified. Flows where perturbations grow as they are convected downstream are known as convectively unstable (CU), while those where the perturbations contaminate the entire flow are known as absolutely unstable (AU). Figure 4 shows how a perturbation will grow in time and space for both types of flows. There are also flows that exhibit a combination of these types of instability at different streamwise locations; these are known as mixed flows.

![Figure 4. Sketch of a perturbation in a convectively (CU) and absolutely (AU) unstable flow.](image-url)

Absolute/convective analysis is only valid if the mean flow is assumed parallel. Most of the time this is not the case and a quasi-parallel base flow analysis is followed. This leads then to the terms
local and global instabilities, which refer to the stability of the local velocity profile and of the entire flow field respectively. An extensive review of absolute/convective stability analysis is given by Huerre et al. in [7], [12] and [13].

In order to find the most dominant frequency of mixed flows, Monkevitz & Sohn [14] suggest distinguishing between two configurations and selecting the frequency criterion appropriately. If the flow contains a region of absolute instability surrounded by two convectively unstable regions (CU-AU-CU), then one should follow Pierrehumbert’s [15] selection criterion. If the flow is absolutely unstable and transitions into a convectively unstable region (AU-CU), then Koch’s [8] approach should be followed.

Pierrehumbert’s analysis of slowly varying baroclinic flows suggests that the dominant frequency in a solid body-CU-AU-CU flow is equal to the real part associated with the maximum absolute growth rate over the entire flow. On the other hand, Koch’s analysis of a solid body-AU-CU flow suggests that the most dominant frequency is the real frequency pertaining to the transition point separating the absolute and convective regions. This is an important point which will affect the analysis of the stability of a liquid jet discussed here.

Global Eigenvalues

As mentioned before, a spatial analysis requires both eigenvalues, \( k \) and \( \eta \), obtained by solving the BVP (equation (12)), to be complex, while the frequency, \( \omega = \alpha c = (\eta/b)(1 - w_c k) \), is real. From the change of variables (11), this is expressed as

\[
\begin{align*}
\omega_r &= \frac{1}{b} (\eta_r - w_c (\eta_r k_r - \eta_i k_i)) \\
\omega_i &= w_c - \frac{\eta_i}{(\eta_r k_i + \eta_i k_r)} = 0.
\end{align*}
\]

A neutral stability point is found by setting the growth rate, \( \alpha_i \), in equation (6) equal to zero (this corresponds to setting \( \eta_i = 0 \)). Solving for this in equations (19) and (20) gives \( \eta_r = 1.598 \), \( k_r = 0.607 \) and \( k_i = 0.0 \). As shown in Figure 5, this point maps as a neutral stability line given by \( \omega_r = 1.598(1 - 0.607 w_c)/b \) in a frequency - streamwise coordinate \((\omega_r - w_c)\) plot. This Figure shows how a disturbance of a given frequency is amplified \((\eta_r < 0)\) or suppressed \((\eta_r > 0)\) as it moves downstream. Furthermore, a line of maximum amplification is obtained numerically from the locus of points for which \( \partial \eta_r / \partial w_c = 0 \). This maximum amplification line suggests that a disturbance close to the nozzle’s exit \((w_c \sim 0.95)\) with a frequency around 0.17 will be the most dominant wave in the flow; since the disturbances are assumed to grow exponentially, this particular wave will outgrow all other unstable waves present in the flow. This wave will then be amplified as it moves downstream until it either transitions to turbulence or saturates, reaches the neutral point again \((w_c \sim 0.12)\), and propagates downstream until it damps out.

![Figure 5. Stability map of frequency against downstream locations \((w_c = 1.0: \text{Nozzle exit. } w_c = 0: \text{Infinity})\).](image)

Note that the mapping in Figure 5 produces a singular point around \( \omega_r \sim 0.17 \) and \( w_c \sim 0.95 \); and, for values where \( \eta_r \) is less than approximately \(-0.7\), the lower branch of the curve turns upward. This type of singularity is discussed in detail in references [9] and [16]. As explained later, the downstream location, \( w_c \), where this singularity takes place, corresponds to the location where an absolutely unstable region transitions to a convectively unstable region.

Local Eigenvalues

Using transformation (11) the local eigenvalues \( \alpha \) and \( c \) are obtained at each streamwise location \( w_c \). The spatial growth rate-\( \alpha_i \) is plotted in Figure 6 against frequency for several values of \( w_c \). This Figure shows that as \( w_c \) increases (regions closer to the nozzle’s exit) the range of unstable frequencies decreases and the single curves eventually split into two separate segments: these segments intersect each other when the group velocity vanishes \((c_g = \partial \omega_r / \partial \alpha_r = 0)\). The streamwise location at which this occurs, \( w_c = w_c^{st} = 0.945 \), signals a transition in the flow from a convectively to an absolutely unstable region as previously explained. A similar analysis has been done by [9] and [17] for a hyperbolic secant squared and a Gaussian wake profile, respectively in a single phase flow.

Mapping the local eigenvalues in the complex frequency plane \((\omega_r - \omega_i)\) reveals a singularity which
changes at each streamwise location. A trace of this singularity as \( w_c \) is varied is plotted in Figure 7. As mentioned by Huerre [7], the crossing of the real axis by this singularity represents a change in the nature of the instability; the flow transitions from being locally convectively unstable (\( AU; \omega_i < 0 \)) to being locally absolutely unstable (\( AU; \omega_i > 0 \)). It is interesting to note that for values of \( w_c < 0.6 \) the singularity no longer appears.

The nature of the instability can also be distinguished by mapping the complex frequency \( \omega_i \), in the complex \( \alpha \) plane and analyzing the location of a saddle point. This kind of analysis has been done by Hultgren et.al. [17] for a Gaussian wake profile.

As mentioned earlier, Koch’s criterion [8] for frequency selection is used to determined the most dominant mode in a flow with mixed absolute and convective instability regions. Following his criterion, the most dominant frequency corresponds to the observed modes at the location of transition from \( AU \) to \( CU \). Therefore, for this particular case, the corresponding modes at transition, \( w_{ct} = 0.945 \), are

\[
\begin{align*}
\omega_r &= 0.177, \quad \omega_i = 0.0 \\
\alpha_r &= 0.301, \quad \alpha_i = -0.295 \\
c_r &= 0.300, \quad c_i = 0.294
\end{align*}
\]

again, \( \omega = \alpha_c \). These results represent the most likely mode that this configuration will amplify. Note that the most unstable local frequency is very close to the global frequency observed by Brennen’s results in Figure 5. The local analysis, however, yields far more insight into the jet’s flow physics: as explained by Huerre [7] for a wake with mixed \( AU/CU \) characteristics, intense self sustained oscillations occur inside an \( AU \) region between a solid body and a \( CU \) region. Following this same train of thought for this liquid jet, the pocket of \( AU \) near the nozzle exit will generate oscillations that will then be convected downstream by the \( CU \) region and, the most unstable mode of these oscillations will corresponds to that observed at the transition location.

**Temporal Analysis**

The solution to Rayleigh’s eigenvalue problem (12) is independent of temporal or spatial analysis; it is the analysis of the eigenvalues that makes the difference. Therefore, the same solution set obtained for the spatial case is used in the following temporal analysis. Except of course, only eigenvalue sets that satisfy \( \eta_i = 0 \) are considered.

As with the local spatial analysis, the eigenvalues are transformed to the original \( \alpha \) and \( c \) via (11). The local temporal eigenvalues obtained at the \( AU/CU \) transition location, \( w_{ct} = 0.945 \), are plotted in Figure 8.

The most unstable mode at this streamwise location, corresponds to that of the largest temporal growth rate, \( \omega_i \). The resulting modes are

\[
\begin{align*}
\omega_r &= 0.159, \quad \omega_i = 0.049 \\
\alpha_r &= 0.264, \quad \alpha_i = 0 \\
c_r &= 0.601, \quad c_i = 0.186
\end{align*}
\]

The temporal growth rate is plotted in Figure 9 for several streamwise, \( w_c \), locations. Note that no singularities or pinching occurs in this plot in contrast with its spatial counterpart, even though there is a negative group velocity region (\( c_g = \partial \omega_r / \partial \alpha_r < 0 \)) in the frequency plot in Figure 8. Figure 9 also shows how the maximum growth rate changes with streamwise location; it increases as \( w_c \) is decreased, it peaks at \( w_c \sim 0.94 \) and it decreases afterwards.

![Figure 6](image6.png)

**Figure 6.** Spatial growth rate at several streamwise locations. \( c_g \) represents the group velocity.

![Figure 7](image7.png)

**Figure 7.** Location of the singularity in the complex frequency plane as a function of downstream location \( \omega_i \). The condition \( \omega_i = 0 \) signals the transition from an \( AU \) to a \( CU \) region.
tion followed by a parallel nozzle of diameter $D_{HT} = 0.635 cm$ and $L/D = 1$ (Figure 10). Laminar flow into the parallel section is ensured by the placement of a honeycomb section upstream of the convergent section as well as a strong favorable pressure gradient ($\Delta P < 0$). The nozzle’s base pressure is set at $345 kN/m^2$ and the reported jet velocity is $U_{HT} = 25.3 m/s$.

Figure 10. Hoyt and Taylor experimental setup [5].

The laminar jet emerging from the nozzle develops small (linear) waves of wavelength $\lambda_{HT} = 0.046 cm$ before transitioning to turbulent flow and breaking up into droplets. According to the authors, the location of these waves fluctuates widely in time. They state, however, that $x^*/D^* = 0.25$ is a representative location.

By assuming that a laminar boundary layer starts to grow at the upstream location of the parallel section, Hoyt and Taylor [5] calculate a momentum thickness, $\delta_t^*$, of 0.0012 cm, which leads to a Reynolds number based on momentum thickness, $Re_{\delta_t^*}$, of 225. They compare their results to Brennen’s [4] theory. However, in order to relate the frequencies, they have to assume a wave velocity equal to the jet speed. Following Yoon’s et al. [6] reproduction of their analysis, a wave with frequency, $f_{Br}^*$, and velocity $U_w^*$ will have a wavelength, $\lambda_{Br}^*$, given by

$$\lambda_{Br}^* = \frac{U_w^*}{f_{Br}^*}. \quad (23)$$

The dimensional frequency, obtained from Brennen’s stability analysis, is

$$f_{Br}^* = \frac{\omega_r}{\delta_t^*}. \quad (24)$$

where $U_j^*$ is the centerline velocity of the liquid jet, $\delta_t^*$ the boundary layer momentum thickness and $\omega_r = 0.177$ the most unstable nondimensional frequency. Therefore, substituting (24) into (23) gives the dimensional, predicted wavelength

$$\lambda_{Br}^* = \frac{2\pi}{\omega_r} \frac{U_{in}^*}{U_j^*} \delta_t^*. \quad (25)$$

Comparision with Experimental Results

A detailed experimental study of water jets discharging in air conducted by Hoyt and Taylor [5], [18] is compared with the above analytical results. The experiment consists of a conical converging section followed by a parallel nozzle of diameter $D_{HT} = 0.635 cm$ and $L/D = 1$ (Figure 10). Laminar flow into the parallel section is ensured by the placement of a honeycomb section upstream of the convergent section as well as a strong favorable pressure gradient ($\Delta P < 0$). The nozzle’s base pressure is set at $345 kN/m^2$ and the reported jet velocity is $U_{HT} = 25.3 m/s$.

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$$\lambda_{Br}^* = \frac{2\pi}{\omega_r} \frac{U_{in}^*}{U_j^*} \delta_t^*. \quad (25)$$
If it is assumed that the ratio $U_{\infty}^*/U_J^*$ is close to unity or that the wave velocity is close to the jet’s velocity, then a wavelength close to the experimentally observed wavelength (0.046cm) is obtained:

$$\lambda_{Br} \sim \frac{2\pi}{\omega_r} \delta_2^* = 0.043cm.$$  \tag{26}

Unfortunately, neither the wave speed predicted from the spatial nor temporal linear stability analysis is close to the jet velocity. (The nondimensional jet velocity is $U_J = 1$.)

In complex analysis, for a wave of the form $e^{i(\alpha x - \omega t)}$, the wave velocity is defined as $u_w = \omega/\alpha$. For a spatial analysis, $\alpha$ corresponds to the growth rate and is therefore not part of the spatial wave itself, and $\omega_i = (\alpha c)_i = 0$. Then, from (21), the spatial wave velocity is

$$U_w(S) = \frac{\omega_r}{\alpha_r} = \left( c_r - \frac{\alpha_i c_i}{\alpha_r} \right) = 0.588.$$  \tag{27}

For a temporal analysis the wave speed is given by the real part of $c$, so from (22)

$$U_w(T) = c_r = 0.601.$$  \tag{28}

Using these wave velocities in equation (26) and the same theoretical frequency obtained by Brennen ($\omega_r = 0.177$) the resulting spatial and temporal wavelengths are 0.025cm and 0.026cm respectively.

To reconcile this apparent discrepancy it should be recognized that the medium in which the perturbation wave is travelling is being accelerated. As a result, this wave stretches as it moves downstream. Wavelength measurements of an arbitrary source wave of fixed frequency will, therefore, differ at each streamwise location.

As a first approximation to account for this non-uniformity of the flow, a Doppler-like analysis \footnote{1} is made. Let $u_w$ represent the wave speed relative to a fixed lab frame of reference, $u_w'$ the wave speed relative to the local fluid, $U_e$ the surface’s edge (medium) velocity, $x_m$ an arbitrary streamwise location where measurements are made, $x_{tr}$ the location where the wave is generated ($AU/CU$ transition location in this analysis) and assume the wave travels at a constant velocity along the fluid, that is $u_w'$ is a constant. Then,

$$u_w'(x_{tr}) = U_e(x_{tr}) = u_w(x_m) - U_e(x_m)$$  \tag{29}

and

$$u_w(x_m) = u_w(x_{tr}) - U_e(x_{tr}) + U_e(x_m).$$  \tag{30}

It is well known that, for a fixed source in a moving medium, the frequency remains unchanged \footnote{2} by the movement of the medium. With this in mind, a measured wavelength can be obtained from

$$\lambda_m = \frac{u_w(x_m)}{f} = \frac{u_w(x_{tr}) - U_e(x_{tr}) + U_e(x_m)}{f}$$

rearranging and defining $\lambda_{tr} = u_w(x_{tr})/f$ the above equation becomes

$$\lambda_m = \lambda_{tr} \left[ 1 + \frac{U_e(x_{tr}) - U_e(x_{tr})}{u_w(x_{tr})} \right]$$  \tag{31}

The assumption that the wave velocity relative to the medium, $u_w'$, is a constant is true only if the parallel base flow assumption is made, which is the case in this analysis. For weakly or strongly varying nonparallel base flows though, this velocity will be a function of its streamwise location and an appropriate analysis should be carried out \footnote{2}.

Hoyt and Taylor’s measurements are reported at a streamwise location of $x^*/D^* = x_m = 0.25$. At this location, the calculated edge velocity, $U_e(x_m) = 0.460$, is obtained from Goldstein’s \footnote{1} analysis (Figure 2). The most unstable wave is generated at the streamwise location where the flow transitions from a region of absolute instability to a region of convective instability. This occurs at $w^e_{tr} = 1 - U_e(x_{tr}) = 0.945$. This corresponds to an edge velocity of $U_e(x_{tr}) = 0.555$. The velocity of this wave, $u_w(x_{tr})$, is calculated to be either 0.588 or 0.601 for a spatial or temporal analysis respectively, see equations (27) and (28). Using the above results and the analysis resulting in equation (32), the most unstable dimensional wavelength predicted from the spatial linear stability is

$$\lambda^*_{m,sp} = \frac{2\pi}{0.301} \left[ 1 + \frac{0.46 - 0.055}{0.588} \right] \delta^*_2 = 0.042cm,$$  \tag{33}

where $\lambda = 2\pi/\alpha_r$, and for the temporal analysis

$$\lambda^*_{m,tmpl} = \frac{2\pi}{0.264} \left[ 1 + \frac{0.46 - 0.055}{0.601} \right] \delta^*_2 = 0.048cm.$$  \tag{34}

As can be seen, both the spatial and temporal theoretical wavelengths are in excellent agreement with the experimental measurements $\lambda^*_{HT} = 0.046cm$ reported by Hoyt and Taylor. Note that the temporal analysis over-predicts the wavelength and the spatial under-predicts it.

It is interesting to mention that the reason the previous comparisons of Brennen’s analysis with the
Figure 11. Comparison of theoretical results from various theories: Kelvin-Helmholtz, Sterling-Sleicher and Brennen’s at various $L/d$’s. Results are for a water jet into air, $d = 1mm$ and $\nu = 1.138 \times 10^{-6}m^2/s$.

Hoyt and Taylor jet were close, i.e. the assumption of $u_w \sim U_J$ worked fine, is because $u_w(0.25) = u_w'(0.25) = (0.588 - 0.055) + 0.46 = 0.993 \sim U_J = 1.0$. That is, at the location the measurements are made ($x^\prime/D^\prime \sim 0.25$), the relative wave velocity plus the medium velocity are in fact very close to the jet velocity. However, at this location, the measured wavenumber, $\alpha_{HT}(2\pi/\lambda^\prime) = 0.164$, does not match its calculated value, $\alpha_{Br} = 0.301$.

Figure 12. Variation of predicted wavelength at several streamwise locations. Results are for a water jet into air, $L/d = 1$, $d = 1mm$ and $\nu = 1.138 \times 10^{-6}m^2/s$.

Several predicted wavelengths from this spatial analysis are compared with predictions from temporal analyses derived from the Kelvin-Helmholtz and Sterling-Sleicher theories [6]. Figure 11 shows results for a $1mm$ diameter water injector discharging in air at several $L/D$’s. The current theory shows a strong influence on $L/D$ while the other theories do not include this parameter; for larger $L/D$, the boundary layer’s momentum thickness at the orifice exit increases, thus increasing the observed most unstable wavelength.

Figure 12 shows how the predicted wavelength changes at several streamwise locations. In general, as the measurements move further downstream, the expected wavelength increases. This result is consistent with the fact that the waves stretch as the velocity at the free surface increases.

Summary
A linear stability analysis of an incompressible liquid jet with a quasi-parallel base flow is presented. The most unstable temporal and spatial instability modes are calculated and successfully compared with experimental measurements. It is found that the assumption of having a wave velocity close to the jet’s velocity (previously used to correlate theory and experiments) is inconsistent with the calculated eigenvalues. This issue is resolved by analyzing surface waves in an accelerating medium, and thus, reconciling both theory and measurements. Figure 13 presents a summary of the results.

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Nomenclature

- $b$ boundary layer half width
- $c$ wave speed, $\omega/\alpha$
- $c_g$ group velocity
- $d$ jet diameter
- $f$ frequency
- $h$ surface wave height
- $k$ global wave number, $(1 - c)/w_c$
- $p$ pressure
- $t$ time
- $U$ base flow streamwise velocity
- $U_e$ edge velocity, $U(x, 0)$
- $U_J$ centerline velocity, $U(x, \infty)$
- $u$ streamwise velocity
- $u_w$ wave velocity with respect to the fixed lab frame
- $V$ base flow normal velocity
- $v$ normal velocity
- $w$ deficit velocity, $1 - U(x, y)$
- $w_c$ deficit edge velocity, $1 - U(x, 0)$
- $x$ streamwise coordinate
- $y$ normal coordinate
- $z$ global normal coordinate, $y/b$
- $\alpha$ wave number
- $\delta_2$ boundary layer momentum thickness
- $\eta$ global wave number, $ab$
\( \lambda \) wavelength

\( \mu \) liquid viscosity

\( \nu \) liquid kinematic viscosity

\( \rho \) liquid density

\( \sigma \) surface tension

\( \omega \) angular frequency

Subscripts

\( Br \) from Brennen’s analysis

\( HT \) from Hoyt and Taylor results

\( i \) imaginary part of a variable

\( j \) subscript

\( m \) measured quantity

\( r \) real part of a variable

\( sp \) from spatial analysis

\( tmp \) from temporal analysis

\( tr \) at AU/CU transition

Superscripts

\( * \) dimensional quantity

\( \cdot \) \( y \)-dependent perturbation function

\( xt \) at AU/CU transition

\( \prime \) measured relative to the moving fluid medium

References


Figure 13. Summary of results near the jet exit.