The Effect of Vapor Bubbles on the Stability of a Liquid Sheet

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Abstract
The study of the breakup of a liquid jet or sheet injected into a gas is essential for fundamentally understanding, as well as for modeling, spray development. In addition to the Kelvin-Helmholtz interface instability, the presence of vapor bubbles in a liquid jet can create the instabilities necessary for the liquid jet or sheet injected into a gas to break up into a dispersion of small liquid droplets. In the present paper, a model attributable to Kogorko [1], Iordaniski [2], and van Wijngaarden [3] is used to investigate the influence of the presence of bubbles on the Kelvin-Helmholtz instability in a liquid. The model accounts for the presence of a gas phase surrounding the moving bubbly liquid sheet. The parameters governing the problem are the Weber number, which describes the ratio of inertial to surface tension forces, the ratio of gas and liquid density, and the speed of sound in the liquid mixture. The sound of speed decreases with bubble concentration in the liquid with a sharp decrease when bubble concentration levels are increased a minute amount from zero. A complex dispersion relation is derived, the analysis of which demonstrates that the presence of bubbles in the liquid does not modify the threshold for the onset of the Kelvin-Helmholtz instability. The instability onset occurs at a zero Weber number, with or without the presence of bubbles. While the instability is initiated at zero wavenumber for a bubble-free liquid, the presence of a minute amount of bubbles in the liquid can drastically alter the critical wavenumber. More precisely, it is found that the critical wavenumber increases with the concentration level of bubbles in the liquid. If the initial size of the droplets in the spray following breakup is correlated with the critical wavelength, our findings imply that the presence of bubbles would cause smaller liquid droplets. One way to produce small vapor bubbles in a liquid jet or sheet is to place a venturi inside the injector so as to produce regions of low pressure that cause the liquid to cavitate.

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Introduction

Many engineering applications depend on the process of a liquid jet breaking into a spray. These applications include injection in engines and turbines, spray painting processes, and ink jet printing. Sprays formed from a liquid round jet or rectangular sheet emanating from a nozzle are particularly germane to combustion applications which inevitably incorporate diffusion-controlled mixing and combustion of liquid fuels. Understanding the physics of a combustor’s fuel spray evolution is essential to understanding the nature of the subsequent combustion process associated with heat release and emissions production for turbines, reciprocating engines, furnaces and boilers.

It is well known that the relative motion at the free surface between a thin liquid sheet and surrounding gas destabilizes the liquid-gas interface via the well known Kelvin-Helmholtz instability. Extensive investigations of these instabilities have improved the understanding of how an initially planar liquid sheet breaks down into droplets under the action of an interfacial shear. A question of long-standing interest to engineers researching liquid sheet atomization is the effect of hydrodynamic conditions on droplet size.

The majority of the theoretical stability investigations of both the planar liquid sheet and the round jet have been linear. (See for example the review and analysis of Senecal et al [4].) These studies reveal that instabilities arise due to competing forces between the stabilizing influence of surface tension and viscosity and the destabilizing effect of the imposed shear stress. The density ratio of the liquid sheet and the ambient gas also affects the threshold values for instability. Linear stability studies demonstrate that the planar sheet exhibits instability as either a sinuous mode or a varicose mode. The former, being antisymmetric, is associated with a constant liquid thickness, while the latter, being symmetrical, is characterized by alternating thin and thick regions of liquid.

Cavitation of a liquid within a nozzle produces vapor bubbles mixed with the liquid jet leaving the nozzle. The addition of vapor bubbles increases the jet’s instability in part due to the two-phase mixture’s compressibility. Brennen [5] demonstrates the extent of this compressibility effect theoretically as illustrated for a water and air mixture. The speed of sound in water at 20°C decreases from around 1500 m/s for pure water to about 25 m/s when a 20% volume fraction of air is mixed with the water at atmospheric pressure. Experimental data from Karplus [6] and Gouse and Brown [7] closely match Brennen’s prediction.

The hydrodynamic stability of fluid flows containing air bubbles, so-called bubbly fluids, has not attracted much attention from the research community due primarily to the complexity that the bubbles add to the model [8]. In this paper we make use of a simple mathematical model developed by Kogarko [1], Iordaniski [2], and van Wijngaarden [3] for an inviscid bubbly fluid to quantify the role of bubbles on the sheet instability.

In this paper, we will analyze the effect of compressibility on the stability fluid sheet composed of a two-phase mixture with a small volume fraction of vapor. Linear equations are formulated that describe the effect that surface tension and momentum have on an inviscid liquid sheet containing a small volume fraction of vapor bubbles. The results illustrate the effect of compressibility on the neutral stability curves as a function of the Weber number, wave number, and density ratio.

Formulation

An inviscid liquid sheet, moving horizontally with a velocity of $U_o$, contains a small volume fraction of spherical bubbles, the concentration of which is conserved. The bubbles are assumed not to interact with each other and move with the fluid with no interphase velocity. The mixture is viewed as a continuous medium with the liquid phase making up its mass and the vapor content accounting for its compressibility [3].

Let $\rho(x,z,t)$, $p(x,z,t)$ and $u(x,z,t)$ denote the average density, pressure, and velocity. Then the fluid flow in the mixture is described by the equations of mass and momentum conservation.

\[
\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho w)}{\partial z} = 0 \tag{1}
\]

\[
\rho \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} \right) = -\frac{\partial p}{\partial x} \tag{2}
\]

\[
\rho \left( \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + w \frac{\partial w}{\partial z} \right) = -\frac{\partial p}{\partial z} \tag{3}
\]
The pressures in the liquid and vapor phases are related by

\[ p_g(R) = p_o \left( \frac{R_o}{R} \right)^{3\gamma} \]  

and

\[ p_g - p = \rho \left( \frac{R \frac{d^2 R}{dt^2} + 3 \left( \frac{dR}{dt} \right)^2}{2} \right) \]

where \( \rho_g \) and \( \rho_l \) are the densities of the vapor and liquid phases, \( p_o \) is the equilibrium pressure, and \( \gamma \) is the polytropic gas exponent. The density of the mixture is given by

\[ \rho = \left( 1 - \frac{4}{3} \pi R^3 N \right) \rho_l + \frac{4}{3} \pi R^3 N \rho_g \]

where \( N(x,z,t) \) and \( R(t) \) represent the number of bubbles per unit volume and the bubble's radius.

The normal stress condition implies

\[ -p + p_a = \sigma \frac{\partial^2 h}{\partial x^2} \]

where \( \sigma \) is the coefficient of surface tension. At the centerline of the sheet, \( z = 0 \), either \( w = 0 \) (symmetric case) or \( \frac{\partial w}{\partial z} = 0 \) (anti-symmetric case) is imposed.

Far away from the sheet, \( u_a \to 0, \ p_a \to 0 \), and \( w_a \to 0 \).

When length, time, velocity, density, and pressure are scaled by \( h_o, h_o / u_o, u_o, \rho_o, \) and \( \rho_o u_o^2 \), the following dimensionless system is obtained,

\[ \frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho w}{\partial z} = 0 \]  

\[ \frac{\partial \rho u}{\partial t} + \frac{\partial u}{\partial x} \rho u + \frac{\partial u}{\partial z} \rho w = -\frac{\partial p}{\partial x} \]  

\[ \frac{\partial \rho w}{\partial t} + \frac{\partial w}{\partial x} \rho u + \frac{\partial w}{\partial z} \rho w = -\frac{\partial p}{\partial z} \]

with boundary conditions,

\[ w = \frac{\partial h}{\partial t} + u_o \frac{\partial h}{\partial x}, \quad w_a = \frac{\partial h}{\partial t} \]  

\[ -p + p_a = \frac{h}{h_o} \frac{\partial^2 h}{\partial x^2} \]  

\[ w = 0 \text{ or } \frac{\partial w}{\partial z} = 0 \]  

\[ u_a \to 0, \ p_a \to 0, \text{ and } w_a \to 0 \text{ as } z \to \infty \]

where \( Q = \frac{\rho_o}{\rho} \) and \( \text{We} = \frac{h_o \rho_o u_o^2}{\sigma} \) are the density ratio and Weber number, respectively.
Gavzilyuk and Teshukov [9] have shown that in the linear regime, the effect of the vapor plane is described by a relationship between the pressure in the liquid phase and the density of the mixture as,

\[ p = \alpha^2 \rho + \beta^2 \left( \frac{\partial}{\partial t} + \frac{\partial}{\partial x} \right)^2 \rho \]  

(22)

where \( \alpha^2 = \frac{a^2}{u_w^2} \), the term, \( a \), being the speed of sound in the bubbly fluid, and \( \beta^2 \) representing the characteristic length scale \( R/A_i \), \( A_i \) being the density of the bubbles’ interfacial area.

On introducing the potential functions, \( \phi \) and \( \phi_a \) defined by

\[ u = \phi_x, \quad w = \phi_z, \quad (\phi_a)_x = u_a, \quad (\phi_a)_z = w_a; \]  

(23)

the governing system reduces to

\[ \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x} + \nabla^2 \phi = 0 \]  

(24)

\[ p = -\frac{\partial \phi}{\partial t} - \frac{\partial \phi}{\partial x} \]  

(25)

\[ p = \alpha^2 \rho + \beta^2 \left( \frac{\partial}{\partial t} + \frac{\partial}{\partial x} \right)^2 \rho \]  

(26)

\[ \nabla^2 \phi_a = 0 \]  

(27)

\[ p_a = -Q \frac{\partial \phi_a}{\partial t} \]  

(28)

\[ \frac{\partial \phi}{\partial z} = \frac{\partial h}{\partial t} + \frac{\partial h}{\partial x} + \frac{\partial \phi}{\partial t} + \frac{\partial \phi}{\partial x} \]  

(29)

\[ -p + p_a = W e^{-1} \frac{\partial^2 h}{\partial x^2} \]  

(30)

\[ \frac{\partial \phi}{\partial z} = 0 \quad \text{(varicose)} \quad \text{or} \quad \phi = 0 \quad \text{(sinuous)} \quad \text{or at} \quad z = 0 \]  

(31)

\[ \phi_a \to 0, \quad \text{and} \quad p_a \to 0 \quad \text{as} \quad z \to \infty \]  

(32)

Equations (25) and (26) imply,

\[ -\alpha^2 \rho - \beta^2 \left( \frac{\partial}{\partial t} + \frac{\partial}{\partial x} \right)^2 \rho = \left( \frac{\partial}{\partial t} + \frac{\partial}{\partial x} \right)^2 \phi \]  

(33)

By operating with \( \left( \frac{\partial}{\partial t} + \frac{\partial}{\partial x} \right) \) on both sides of equation (33) and on using equation (24), we find,

\[ \alpha^2 \nabla^2 \phi + \beta^2 \left( \frac{\partial}{\partial t} + \frac{\partial}{\partial x} \right)^2 \nabla^2 \phi = \left( \frac{\partial}{\partial t} + \frac{\partial}{\partial x} \right)^2 \phi \]  

(34)

The analysis of the system consisting of equations (34), (27), and (28) and boundary conditions given by equations (29) – (32) using the Fourier transform, \( \phi = \hat{\phi}(z) \exp(i \omega t + ikz) \), yields the following dispersion relation,

\[ k(\omega + ik)^2 \coth(l) + Q l \omega^2 = -l R k^3 \]  

(35)

for the varicose mode, and

\[ k(\omega + ik)^2 \tanh(l) + Q l \omega^2 = -l R k^3 \]  

(36)

for the sinuous mode, where

\[ l = \left[ k^2 + \frac{(\omega + ik)^2}{\alpha^2 + \beta^2(\omega + ik)^2} \right]^{1/2} \]  

(37)

and \( \omega \) and \( k \) are the growth rate and the wavenumber.

**Results**

Equations (35) and (37) describe the relation between the growth rate, the inverse Weber number, the wave number, the density ratio, and the nondimensionalized speed of sound for the varicose mode of liquid sheet deformation. Equations (36) and (37) describe the relation between these variables for the sinuous mode of liquid sheet deformation. In general, the complex equations are difficult to analyze; however, we have been able to determine the neutral stability curves as functions of the wave number and the Weber number for varying values of the nondimensionalized speed of sound \( \alpha \).

The term \( (\omega + ik)^2 / \left[ \alpha^2 + \beta^2(\omega + ik)^2 \right] \) in Eq. (37) determines the influence of the presence of bubbles on the onset of instability. The stability characteristics can be determined directly by solving the dispersion relation, equation (35), for the growth rate \( \omega \) as function of the wavenumber \( k \) for fixed values of the Weber number and the density ratio \( Q \). This approach, how-
ever, does not yield a quantitative assessment of the wavelength of the instability.

At the onset of instability, the real part of $\omega$ is zero. Thus, on setting $\omega = ib$ in equation (35), we obtain

$$We = \left[ \frac{(b + k)^2}{k^2} \frac{1}{l^*} \coth(l^*) + \frac{Qb^2}{k^3} \right]^{-1}$$

where

$$l^* = \left[ k^2 - \frac{(b + k)^2}{\alpha^2 - \beta^2(b + k)^2} \right]^{1/2}$$

for the varicose mode, with a similar equation for the sinuous mode except that the hyperbolic cotangent is replaced with the hyperbolic tangent.

Figure 1 illustrates the division of the $\alpha$-$k$ space into physical and nonphysical regions when $E$ is set to 1.0, $b$ to 1.0 and $Q$ to 0.001. These parameters describe a sheet of liquid water moving through air.

A plot of equation (38) versus $k$, for a set of parameters that is physically realizable according to figure (1), is shown in figure (2).

Equations (38) for the varicose mode and its equivalent for the sinuous mode can be readily solved to yield neutral stability curves for various values of $\alpha$ as a function of the wave and Weber numbers.

Figure 2 illustrates the neutral stability curves that describe the varicose mode for the parameter combination used to plot in figure 1. The solid line describes a liquid sheet containing no vapor bubbles. The interaction between the purely liquid sheet and a gas has been described by Senecal et al. [4] who explain that as the wavenumber decreases, the Weber number for sheet instability decreases to the limit of zero Weber number for zero wavenumber.

As explained earlier, for low concentration of bubbles the increase of bubble concentration to the liquid sheet increases the mixture’s compressibility which subsequently decreases the sound speed in the mixture. In figure 2, therefore, smaller values of alpha correspond to larger bubble concentrations. In general the effect of increasing bubble concentration seems to lower neutral stability curve. A small bubble concentration necessary to yield a nondimensional sound of speed to 6.1188 causes the neutral stability curve to fall slightly below the “no bubbles”, or purely liquid, case. The smallest value of alpha studied indicates that when $\alpha$ decreases to 2.8402, the flow is unstable until the wavenumber reaches 0.9 and can be stable near this wavenumber only for Weber numbers less than 0.004.
two-phase such that alpha is 3.77, the sheet in either mode is less stable than the purely liquid sheet. The effect of bubble addition, however, is considerably greater when the sheet’s interface is varicose.

Finally, the effect of the density ratio \( Q \) was studied. When \( Q \) was increased by an order of magnitude over that used for the plots in figures 2 and 3, no noticeable effect of the neutral stability curves could be discerned either for the varicose or the sinusuous modes.

Experimental Investigation

Two flat-spray injectors have been constructed from transparent material. The two injectors have the same overall internal dimensions and slot geometry, but one has an internal venturi designed to create a region of high speed flow. The venturi has been designed such that the associated decrease in pressure caused by the flow acceleration will create cavitation and, subsequently, a bubbly liquid sheet. The objective is to determine what visible effect the presence of internal cavitation would have on the break-up length of the jet leaving the slot. Figure 5 shows a schematic of the injector with the internal venturi. Assuming incompressible flow in the constant width slot nozzle, the fluid velocity anywhere in the slot can be related to the velocity at the venturi throat, \( V_{V} \), and the venturi contraction ratio defined using the nozzle heights, \( L/L_V \):

\[
V_{v} = \frac{L}{L_V} V \quad (40)
\]

The saturation pressure of water at room temperature is approximately 2.5 kPa, so the pressure needs to be reduced to at least this value to induce cavitation. If viscous losses are neglected, the Bernoulli equation can be used to relate the head added by the pump, \( H_P \), to the required velocity at the venturi throat to reach the desired cavitation pressure:

\[
V_{v} = \left(2 \left(\frac{P_{atm} - P_{sat}}{\rho} \right) + 2 g H_P \right)^{\frac{1}{2}} \quad (41)
\]

The same analysis can be applied from the pump to the nozzle exit to determine the velocity at the exit, which is at atmospheric pressure:

\[
V_E = \left(2 g H_P \right)^{\frac{1}{2}} \quad (42)
\]

These three equations can now be used to find the required pump head and venturi contraction ratio to ap-
density approach cavitation for a desired exit velocity. Without a nucleation site, liquid will not boil unless the state is on the superheated side of the saturation conditions. Microscopic particles in the injected liquid would likely be present to serve this purpose, but to account for this and the inviscid assumption in the above analysis, the contraction ratio may need to be somewhat greater than the calculations indicate. For this experiment, a nominal exit velocity of 20 m/s has been chosen. Using this exit velocity, these equations indicate that the supply pump should add 21 m of head, and the contraction ratio should be at least 1.22. To insure cavitation, this ratio has been increased to 2.85.

Subscripts
- \( a \) gas surrounding sheet
- \( g \) vapor
- \( l \) liquid
- \( o \) equilibrium

Conclusions

The breakup of a bubbly inviscid liquid sheet has been analyzed. A bubbly liquid model has been considered for which the stability analysis is mathematically tractable and which contains the main physical ingredients caused by the presence of the bubbles. For instance, the coupled effect of the volume fraction of bubbles and compressibility are included. The speed of sound in the bubbly mixture appears as an important parameter in the model. The results indicate that the presence of the bubbles does not alter the threshold for the onset of instability. Within the model’s assumptions, the sheet is unstable at zero wavenumber, with or without the presence of bubbles. While in the pure liquid case instability sets in at zero wavenumber; however, any level of bubble concentration, albeit very small, modifies the critical wavenumber. If the size of the droplets upon sheet breakup is correlated with the critical wavelength, then the presence of bubbles would act in such a way as to produce smaller size liquid droplets.

Nomenclature

- \( a \) speed of sound
- \( A_i \) density of bubbles’ interface area
- \( b \) complex part of growth function
- \( h \) location of interface in \( z \)-direction
- \( k \) wave number
- \( l \) expression defined in paper
- \( l^* \) expression defined in paper
- \( N \) number of bubbles per volume
- \( p \) pressure
- \( Q \) ratio of gas density to sheet density
- \( R \) bubble radius
- \( t \) time
- \( u, w \) velocity components
- \( x, z \) spatial coordinates
- \( w \) \( y \)-velocity component
- \( \rho \) density
- \( \sigma \) surface tension
- \( \omega \) growth function (complex)

References
